Quantum query complexity of entropy estimation

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Outline

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Motivation

Why do I come across this problem?

- Quantum property testing [e.g., survey [Montanaro, de Wolf]] (Quantum testers of classical properties !)
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- Testing of *distributional properties*. A well-motivated branch in the classical property testing literature !! also connected to learning problems.
Motivation

Why do I come across this problem?

- Quantum property testing [e.g., survey [Montanaro, de Wolf]] (Quantum testers of classical properties !)
- Testing of distributional properties. A well-motivated branch in the classical property testing literature !! also connected to learning problems.
- Objects are distributions, rather than boolean functions. Might bring new insights or call for new techniques!
Entropies

Given any distribution \( p \) over a discrete set \( X \), the Shannon entropy of this distribution \( p \) is defined by

\[
H(p) := \sum_{x \in X : p(x) > 0} -p_x \log p_x.
\]
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$$H(p) := \sum_{x \in X: p(x) > 0} -p_x \log p_x.$$  

One important generalization of Shannon entropy is the Rényi entropy of order $\alpha$, denoted $H_\alpha(p)$, which is defined by

$$H_\alpha(p) = \begin{cases} \frac{1}{1-\alpha} \log \sum_{x \in X} p_x^\alpha, & \text{when } \alpha \neq 1, \\ H(p), & \text{when } \alpha = 1. \end{cases}$$
Problem Statement

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A natural question: Given access to samples obtained by taking independent draws from $p$, determine the necessary number of independent draws to estimate $H(p)$ or $H_{\alpha}(p)$ within error $\epsilon$, with high probability.

Motivations: this is a theoretically appealing topic with intimate connections to statistics, information theory, learning theory, and algorithm design.
Our Question

Main Question: is there any quantum speed-up of estimation of Shannon and Rényi entropies?

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The first research paper on this topic is by Bravyi, Harrow, and Hassidim [BHH 11], where they have discovered quantum speed-ups of testing uniformity, orthogonality, and statistical difference on unknown distributions, followed by Chakraborty et al. [CFMdW 10].
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Classical results

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If $\epsilon$ is a constant, then the classical query complexity

- for Shannon entropy [VV 11], [JVHW 15]: $\Theta(\frac{n}{\log n})$.
- for Rényi entropy [AOST 17]:
  \[
  \begin{cases}
  O(\frac{n^{\frac{1}{\alpha}}}{\log n}) \text{ and } \Omega(n^{\frac{1}{\alpha} - o(1)}), & \text{when } 0 < \alpha < 1. \\
  O(\frac{n}{\log n}) \text{ and } \Omega(n^{1-o(1)}), & \text{when } \alpha > 1, \alpha \notin \mathbb{N}.
  \end{cases}
  \]
  \[
  \Theta(n^{1-\frac{1}{\alpha}}), & \text{when } \alpha > 1, \alpha \in \mathbb{N}.
  \]


Quantum results

Our results:

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>classical bounds</th>
<th>quantum bounds (this talk)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0 &lt; \alpha &lt; 1$</td>
<td>$O(n^{\frac{1}{\alpha \log n}}), \Omega(n^{\frac{1}{\alpha} - o(1)})$ [AOST 17]</td>
<td>$~O(n^{\frac{1}{\alpha} - \frac{1}{2}}), \Omega(\max{n^{\frac{1}{\alpha} - o(1)}, n^{\frac{1}{3}}})$</td>
</tr>
<tr>
<td>$\alpha = 1$</td>
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<td>$~O(\sqrt{n}), \Omega(n^{\frac{1}{3}})$</td>
</tr>
<tr>
<td>$\alpha &gt; 1, \alpha \notin \mathbb{N}$</td>
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<td>$~O(n^{1-\frac{1}{2\alpha}}), \Omega(\max{n^{\frac{1}{3}}, \Omega(n^{\frac{1}{2} - \frac{1}{2\alpha}})})$</td>
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<td>$~O(n^{\nu(1-1/\alpha)}), \Omega(n^{\frac{1}{2} - \frac{1}{2\alpha}}), \nu &lt; 3/4$</td>
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<tr>
<td>$\alpha = \infty$</td>
<td>$\Theta(n^{\frac{1}{\log n}})$ [VV 11]</td>
<td>$~O(Q([\log n]\text{-distinctness})), \Omega(\sqrt{n})$</td>
</tr>
</tbody>
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Table 1: Summary of classical and quantum query complexity of $H_\alpha(p)$ for $\alpha > 0$, assuming $\epsilon = \Theta(1)$. 
Quantum results

Figure 1: Visualization of classical and quantum query complexity of $H_\alpha(p)$. The $x$-axis represents $\alpha$ and the $y$-axis represents the exponent of $n$. Red curves and points represent quantum upper bounds. Green curves and points represent classical tight bounds. The Magenta curve represents quantum lower bounds.
Quantum results: in my April talk at MSR

Figure 2: Visualization of classical and quantum query complexity of $H_\alpha(p)$. The $x$-axis represents $\alpha$ and the $y$-axis represents the exponent of $n$. Red curves and points represent quantum upper bounds. Green curves and points represent classical tight bounds. The Magenta curve represents quantum lower bounds.
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Sample vs Query model

Ref. [BHH 11] models any discrete distribution \( p_i \) on \([n]\) by an oracle \( O_p : [S] \rightarrow [n] \). Any probability \( p_i (i \in [n]) \) is thus proportional to the size of pre-image of \( i \) under \( O_p \):

\[
p_i = \left| \{ s \in [S] : O_p(s) = i \} \right|
\]

\( S \forall i \in [n] \).

If one samples \( s \) uniformly from \([S]\) and outputs \( O_p(s) \), then one obtains a sample drawn from distribution \( p \).

It is shown in [BHH 11] that the query complexity of the oracle model above and the sample complexity of independent samples are in fact equivalent classically.
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It is shown in [BHH 11] that the query complexity of the oracle model above and the sample complexity of independent samples are in fact equivalent classically.
Quantum Query Model

Quantumly, $O_p$ is transformed into a unitary operator $\hat{O}_p$ acting on $\mathbb{C}^S \otimes \mathbb{C}^{n+1}$ such that

$$\hat{O}_p |s\rangle |0\rangle = |s\rangle |O_p(s)\rangle \quad \forall s \in [S].$$

This is more powerful than $O_p$ because we may take a superposition of states as an input.
Roadmap of quantum entropy estimation: for all $\alpha > 0$

- Span programs
- Simulated annealing
- Amplitude amplification
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- Learning graph for $k$-distinctness
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High-level Framework

A general distribution property estimation problem:

Given a discrete distribution $p = (p_i)_{i=1}^n$ on $[n]$ and a function $f : (0, 1] \to \mathbb{R}$, estimate $F(p) := \sum_{i \in [n]} p_i f(p_i)$ with small additive or multiplicative error with high success probability.
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If \( f(x) = -\log x \), \( F(p) \) is the Shannon entropy \( H(p) \) (additive error); if \( f(x) = x^{\alpha-1} \) for some \( \alpha > 0, \alpha \neq 1 \), \( H_\alpha(p) \propto \log F(p) \) (multiplicative error).
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Inspired by BHH, we formulate a framework for approximating $F(p)$. 
## High-level Framework

**Algorithm:** Estimate $F(p) = \sum_i p_i f(p_i)$.

1. Set $l, M \in \mathbb{N}$;
2. **Regard the following subroutine as $A$:**
   - Draw a sample $i \in [n]$ according to $p$;
   - Use **amplitude estimation** with $M$ queries to obtain an estimation $\tilde{p}_i$ of $p_i$;
   - Output $X = f(\tilde{p}_i)$;
3. Use $A$ for $l$ times in “quantum speedup of Chebyshev’s inequality” and outputs an estimation $\tilde{F}(\tilde{p})$ of $F(p)$;
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Cons: very limited quantum speedup with only the basic framework.
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Quantum algorithm for Shannon entropy estimation

Two **new** ingredients (c.f. BHH) when \( f(p) = -\log p \)
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- Montanaro’s quantum speedup of Monte Carlo methods. Let $A$ be a quantum algorithm outputting $X$ such that $\text{Var}(X) \leq \sigma^2$. For $\epsilon$ s.t. $0 < \epsilon < 4\sigma$, by using $\tilde{O}(\sigma/\epsilon)$ times of $A$ and $A^{-1}$, one can output estimate $\tilde{E}(X)$ of $E(X)$ s.t.

$$\Pr[|\tilde{E}(X) - E(X)| \geq \epsilon] \leq 1/4.$$ 

Classically, one needs to use $\Theta(\sigma^2/\epsilon^2)$ times of $A$. 

a fine-tuned analysis to bound the value of $\log(1/p)$ when $p \to 0$. Analyze the full distribution of amplitude amplification.  

Complexity: Classical $\Theta(n/\log(n))$ vs Quantum $\tilde{O}(\sqrt{n})$. 
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Issue w/ the **basic** framework for Rényi entropy

*multiplicative* errors (Montanaro’s algorithm yields no speedup in the worst case) $\rightarrow$ quantum advantage when $1/2 < \alpha < 2$. 
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*Use $P_{\alpha_2}$ to estimate $[a, b]$ for $P_{\alpha_1}$ where $\alpha_2$ is closer to 1.*
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*Use* $P_{\alpha_2}$ *to estimate* $[a, b]$ *for* $P_{\alpha_1}$ *where* $\alpha_2$ *is closer to 1.*

- Recursively solve the $P_{\alpha_2}$ case until $\alpha_2 \approx 1$ where a speed-up is already known.
Quantum algorithm for $\alpha$-Rényi entropy: $\alpha \notin \mathbb{N}$

Recursively call roughly $O(\log(n))$ times until $\alpha \approx 1$

- $\alpha > 1$: $\alpha \to \alpha(1 + \frac{1}{\log n})^{-1} \to \alpha(1 + \frac{1}{\log n})^{-2} \cdots$;
- $0 < \alpha < 1$: $\alpha \to \alpha(1 - \frac{1}{\log n})^{-1} \to \alpha(1 - \frac{1}{\log n})^{-2} \cdots$. 

Similarity and difference: cooling schedules in simulated annealing, volume estimation

$\triangleright$ multi-section, multiplicative factors, similar design principle.

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**Frequency moments**

A sequence $a_1, \ldots, a_M \in [n]$ are given with the occurrences of $1, \ldots, n$ to be $m_1, \ldots, m_n$ respectively. You are asked to give a good approximation of $F_k = \sum_{i \in [n]} m_i^k$ for $k \in \mathbb{N}$. 

Empirical Estimation

Key observation: $\sum_{i \in [n]} p_{\alpha i} \approx \sum_{i \in [n]} \left(\frac{m_i}{M}\right)^k = F_k / M^k$.

Note: this is not the best classical estimator. Why?
Quantum algorithm for $\alpha$-Rényi entropy: $\alpha \geq 2 \in \mathbb{N}$

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Want to use this empirical estimator in the query model !!

Issues and Solutions

- **Issue 1**: How to generate $M$ samples? cannot query $M$ times (the same complexity as classical).

Key observation: treat our quantum oracle $O_p$ as a sequence of $S$ samples. The $\alpha$-frequency moment of $O_p(1), \ldots, O_p(S) = \sum_{i \in [n]} p^\alpha_i$. Roughly replace $S$ by $\alpha n$ in the algorithm and redo the analysis.
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- **Issue 2**: quantum algorithm for $\alpha$-frequency moment with query complexity (e.g., $o(n^{3(1-\frac{1}{\alpha})})$ [Montanaro 16]) where the ”$n$” is actually ”$S$” in this context.
Quantum algorithm for $\alpha$-Rényi entropy: $\alpha \geq 2 \in \mathbb{N}$

Want to use this empirical estimator in the query model!!

Issues and Solutions

- **Issue 1**: How to generate $M$ samples? cannot query $M$ times (the same complexity as classical).
  
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  **Key observation**: Roughly replace $S$ by $\alpha n$ in the algorithm and redo the analysis.
Roadmap of quantum entropy estimation: for all $\alpha > 0$

- Span programs
  - Learning graph for $k$-distinctness
    - Min-entropy estimation
  - Simulated annealing
    - Chebyshev cooling
      - $\alpha$-Rényi entropy estimation ($\alpha \in \mathbb{N}$)
    - Quantum speedup of Chebyshev’s inequality
      - $\alpha$-Rényi entropy estimation ($\alpha \notin \mathbb{N}$)
    - Quantum counting
      - Shannon entropy estimation
  - Amplitude amplification
  - Phase estimation
The min-entropy (i.e., $\alpha = +\infty$) case: find the $\max_i p_i$

How about using integer $\alpha$ algorithm?

Intuitively, exists some $\alpha$ s.t., $H_\alpha(p)$ (sub-linear for any $\alpha$) is a good enough approximation of $H_{\min}(p)$. 

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- ”Poissonized sampling”: the numbers of occurrences of $i$th point are pair-wise independent Poisson distributions, parameterized by a guess value $\lambda$ and $p_i$, $i \in [n]$. 
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- ”Poissonized sampling”: the numbers of occurrences of $i$th point are pair-wise independent Poisson distributions, parameterized by a guess value $\lambda$ and $p_i$, $i \in [n]$.
- log($n$) is a natural cut-off threshold for Poisson distribution.
- When $\lambda \cdot \max_i p_i$ passes this threshold, a log($n$)-collision can be found w.h.p.. Otherwise, update $\lambda$ and try again.
Quantum lower bounds

Any quantum algorithm that approximates $\alpha$-Rényi entropy of a discrete distribution on $[n]$ with success probability at least $2/3$ must use

- $\Omega(n^{\frac{1}{7\alpha}-o(1)})$ quantum queries when $0 < \alpha < \frac{3}{7}$.
- $\Omega(n^{\frac{1}{3}})$ quantum queries when $\frac{3}{7} \leq \alpha \leq 3$.
- $\Omega(n^{\frac{1}{2}-\frac{1}{2\alpha}})$ quantum queries when $\alpha \geq 3$.
- $\Omega(\sqrt{n})$ quantum queries when $\alpha = +\infty$. 

Techniques:

- Reductions to the collision, Hamming weight, symmetry function problems.
- The polynomial method inspired by the collision lower bound for a better error dependence.
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Outline

Motivation and Problem Statements

Main Results

Techniques

Open Questions and On-going Work
Questions

$k$-distinctness for super-constant $k$?

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At a high level, we want testers with correct expectations but small variances. Many techniques are proposed classically, and so different from ours.
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At a high level, we want testers with correct expectations but small variances. Many techniques are proposed classically, and so different from ours.

*Can we leverage the design principle of classical testers?*
Thank You!!

Q & A
Classical estimators for Shannon entropy

A first choice: empirical estimator. If we take $M$ samples and occurrences of $1, \ldots, n$ are $m_1, \ldots, m_n$ respectively, then empirically the distribution is $(m_1/M, \ldots, m_n/M)$, and

$$H_{\text{emp}}(p) = -\sum_{i \in [n]} m_i/M \log m_i/M.$$ 

To approximate $H(p)$ within error $\epsilon$ with high probability, need $M = \Theta(n\epsilon^{-2})$.

Not that bad compared to the best estimator, where $M = \Theta(n\epsilon^{-2}\log n)$. 

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Construction of the best estimator:
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- [VV 11]: A very clever (but complicated) application of linear programming under Poissonized samples (will explain later)
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Construction of the best estimator:

- [VV 11]: A very clever (but complicated) application of linear programming under Poissonized samples (will explain later)
- [JVHW 15]: polynomial approximation
Classical estimators for Rényi entropy

Still, we can first consider the empirical estimator:

$$H_{\alpha, \text{emp}}(p) = \frac{1}{1-\alpha} \log \sum_{i \in [n]} (m_i M)^\alpha.$$ 

In [AOST 17], it is shown that the query complexity of the empirical estimator is:

$$\Theta(n^{1/\alpha}) \quad \text{when } 0 < \alpha < 1,$$

$$\Theta(n) \quad \text{when } \alpha > 1.$$ 

Recall that the classical query complexity of Rényi entropy:

$$\begin{cases} 
O(n^{1/\alpha} \log n) \quad \text{and } \Omega(n^{1/\alpha - o(1)}) \\
O(n^{1/2}) \quad \text{and } \Omega(n^{1/2 - o(1)}) \\
\Theta(n^{1/\alpha - 1}) \quad \text{when } \alpha > 1, \alpha \in \mathbb{N}
\end{cases}$$
Classical estimators for Rényi entropy

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O\left(\frac{n^{\frac{1}{\alpha}}}{\log n}\right) \text{ and } \Omega(n^{\frac{1}{\alpha} - o(1)}), & \quad \text{when } 0 < \alpha < 1. \\
O\left(\frac{n}{\log n}\right) \text{ and } \Omega(n^{1-o(1)}), & \quad \text{when } \alpha > 1, \alpha \notin \mathbb{N}. \\
\Theta(n^{1-\frac{1}{\alpha}}), & \quad \text{when } \alpha > 1, \alpha \in \mathbb{N}.
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Main difference happens only when $\alpha > 1, \alpha \in \mathbb{N}$. 
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This is not a minor point. In particular, 2-Rényi entropy (also known as the collision entropy) has classical query complexity $\Theta(\sqrt{n})$, which is much less than that of Shannon entropy ($\Theta(\frac{n}{\log n})$).
Main difference happens only when $\alpha > 1, \alpha \in \mathbb{N}$.

This is not a minor point. In particular, 2-Rényi entropy (also known as the *collision entropy*) has classical query complexity $\Theta(\sqrt{n})$, which is much less than that of Shannon entropy ($\Theta(\frac{n}{\log n})$).

Collision entropy gives a rough estimation of Shannon entropy, and it measures the quality of random number generators and key derivation in cryptographic applications.
When $\alpha > 1$ and $\alpha \in \mathbb{N}$, [AOST 17] proposes the following “bias-corrected” estimator:

$$H_{\alpha,\text{bias}}(p) = \frac{1}{1 - \alpha} \log \sum_{i \in [n]} \frac{m_i(m_i - 1) \cdots (m_i - \alpha + 1)}{M^\alpha}.$$ 

This is actually the best estimator with query complexity $\Theta(n^{1 - \frac{1}{\alpha}})$.
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$$H_{\alpha,bias}(p) = \frac{1}{1 - \alpha} \log \sum_{i \in [n]} \frac{m_i(m_i - 1) \cdots (m_i - \alpha + 1)}{M^\alpha}.$$ 

This is actually the best estimator with query complexity $\Theta(n^{1-\frac{1}{\alpha}})$. The analysis is mainly based on a “Possionization” technique.
An intuition of Possionization

When you are taking $M$ samples where $M$ is fixed, then the occurrences of 1, ..., $n$ are not independent. But if you take $M \sim \text{Poi}(\lambda)$, then the occurrence of $i$ is $m_i \sim \text{Poi}(\lambda p_i)$, and all $m_i$ are pairwise independent.

Moreover, with high probability $\lambda/2 \leq M \leq 2\lambda$, so Possionization does not influence query complexity up to a constant.

A key reason to use bias-corrected estimator: if a random variable $X \sim \text{Poi}(\lambda)$, then $\forall r \in \mathbb{N}$,

$$E[X^r] = \lambda^r.$$ 

This property helps a lot for the analysis in [AOST 17] based on Chernoff-Hoeffding inequality.
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References

G. Valiant and P. Valiant.
Estimating the unseen: an n/log (n)-sample estimator for entropy and support size, shown optimal via new CLTs.

J. Jiao, K. Venkat, Y. Han, and T. Weissman.
Minimax estimation of functionals of discrete distributions.

Estimating Rényi entropy of discrete distributions.

S. Bravyi, A. W. Harrow, and A. Hassidim.
Quantum algorithms for testing properties of distributions..
*IEEE Transactions on Information Theory, 57.6 (2011): 3971-3981.*

A. Montanaro.
Quantum speedup of Monte Carlo methods.