Math Review:
Functions, Distributions, Vectors, and Matrices

Jordan Boyd-Graber

University of Maryland

Fall 2020

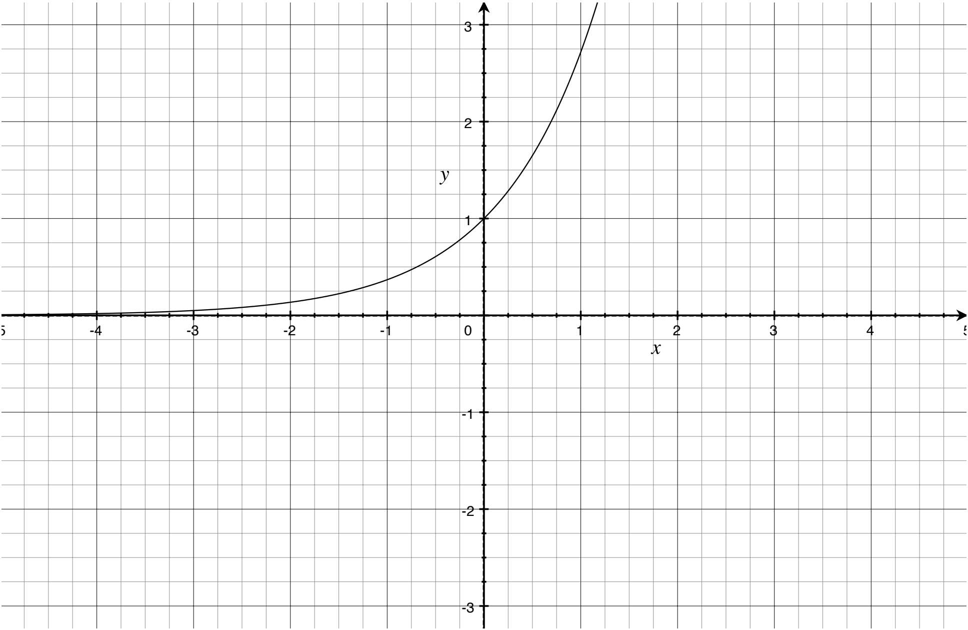
Function Notation

- Take a number a double it
- Mathematical notation

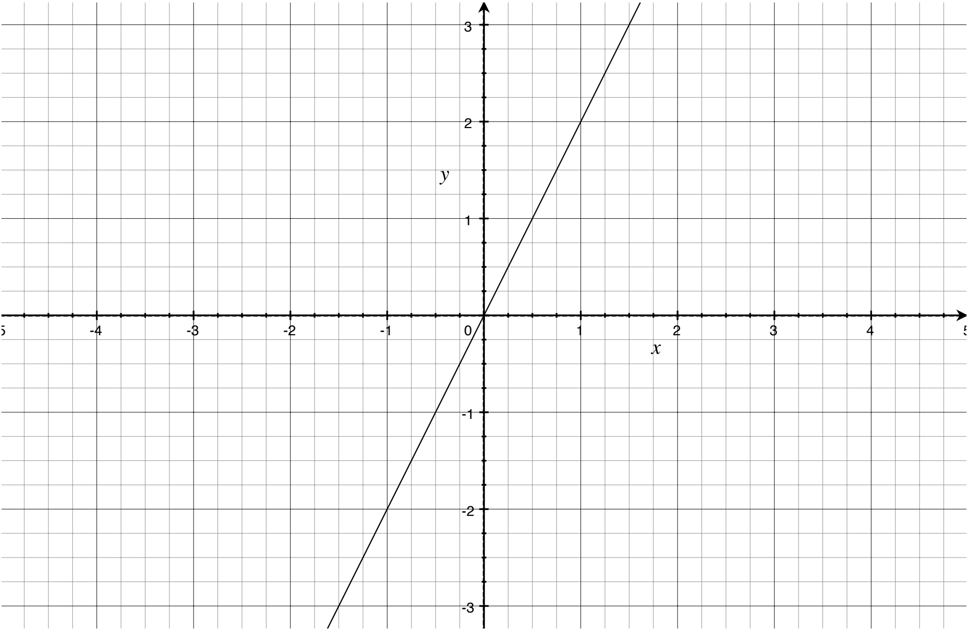
$$f(x) = 2x \tag{1}$$

- Python notation

```
def double(x):
    return 2 * x
```



$$f(x) = \exp(x)$$



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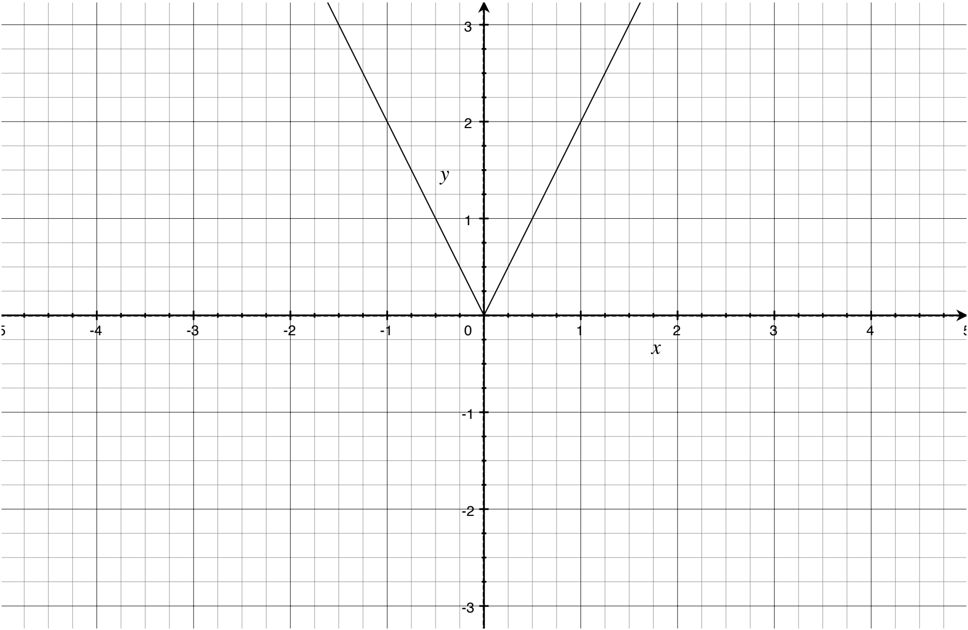
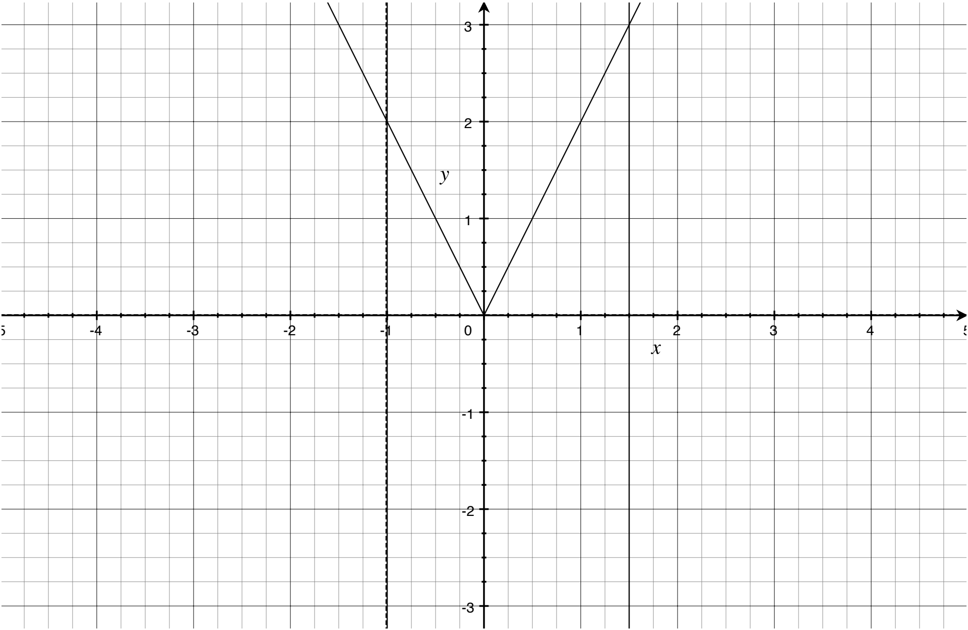


$$f(x) = |x|$$



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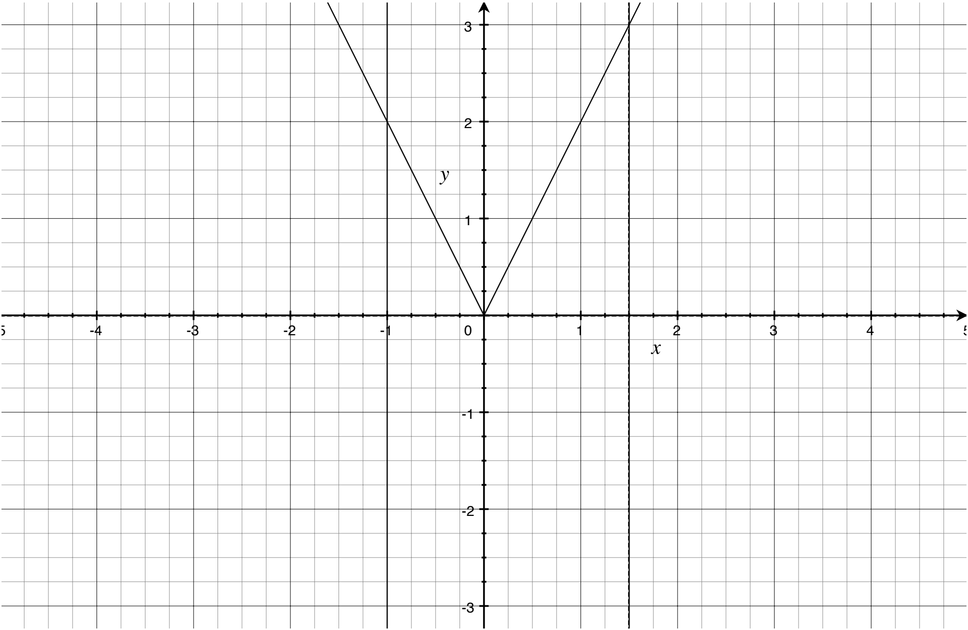

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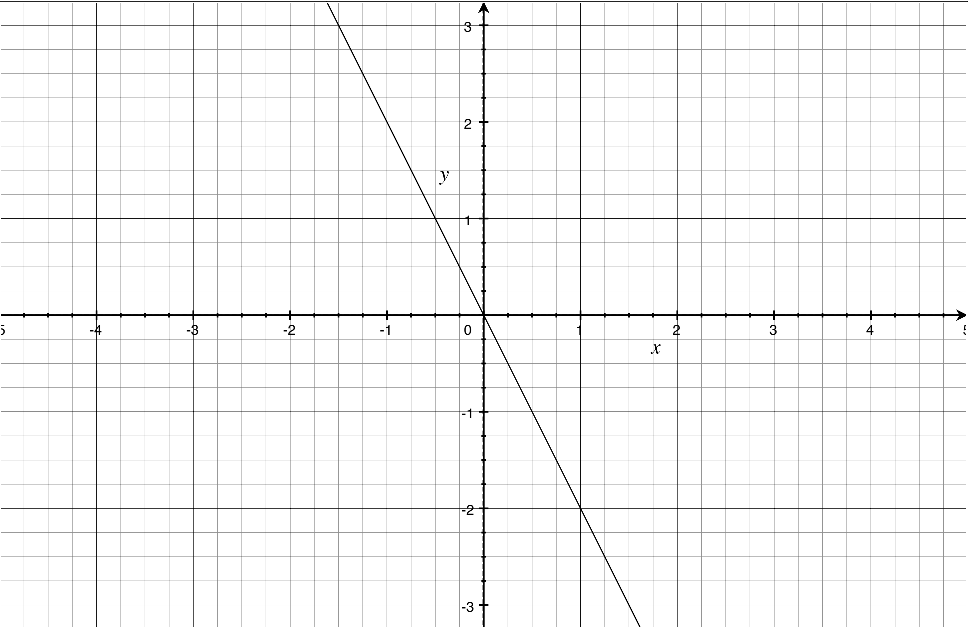
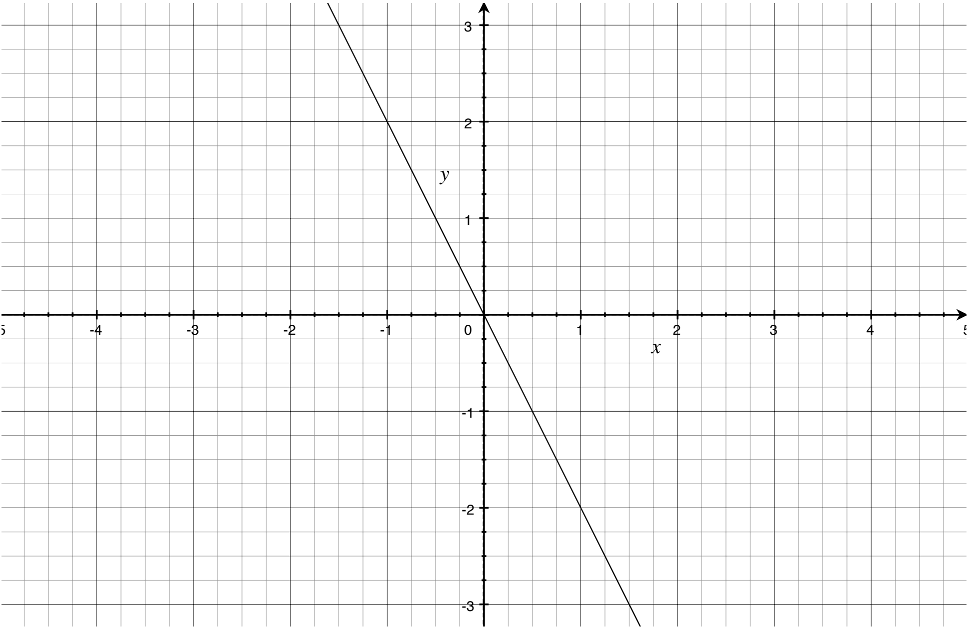
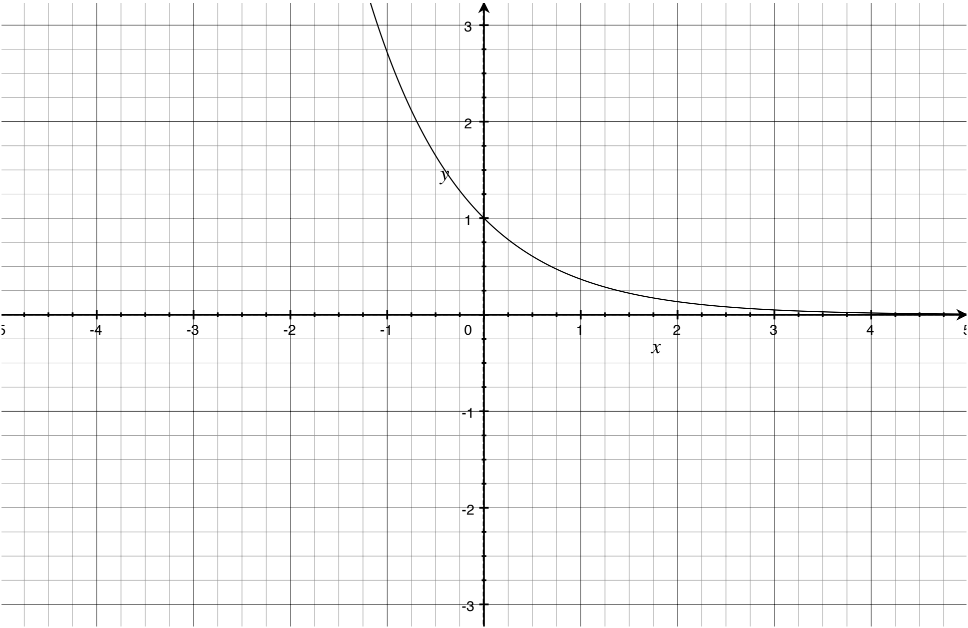


y
x
3
2
1
0
-1
-2
-3
-4
-3
-2
-1
1
2
3
4



$$f(x) = 2(-x)$$



$$f(x) = \exp(-x)$$

Combining functions

$$f(x) = \exp(-x) \tag{2}$$

$$g(x) = 1 + x \tag{3}$$

$$h(x) = \frac{1}{x} \tag{4}$$

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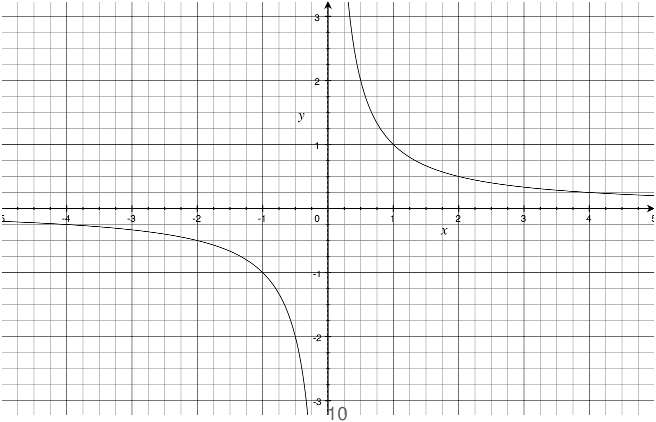

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$$l(x) = g(f(x)) = \frac{1}{\exp(-x)} \tag{5}$$

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```
from math import exp

def neg_exp(x):
  return exp(-x)
def composition(x):
  return 1.0 / neg_exp(x)
```

Properties of the Exponential (and log) Function

$$\exp(a+b) = exp(a)exp(b) \tag{6}$$

$$\log(a+b) = \log(a)\log(b) \tag{7}$$

$$\log(a^b) = b \cdot \log(a) \tag{8}$$

Composition didn't do as much as we thought!

$$
\begin{aligned}
I(x) &= g(f(x)) && (9)\\
&= \frac{1}{\exp(-x)} && (10)\\
&= \frac{1}{\exp(x)^{-1}} && (11)\\
&= \frac{1}{\frac{1}{\exp(x)}} && (12)\\
&= \exp x && (13)
\end{aligned}
$$

Logistic Function

$$f(x) = \exp(-x) \tag{14}$$
$$g(x) = 1 + x \tag{15}$$
$$h(x) = \frac{1}{x} \tag{16}$$

Putting them together:

$$\tag{17}$$

Logistic Function

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$$l(x) = h(g(f(x))) \tag{17}$$

$$\tag{18}$$

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& && (19)
\end{aligned}
$$

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$$= \frac{1}{1 + \exp(-x)} \tag{20}$$

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Engineering rationale behind probabilities

- Encoding uncertainty
  - Data are variables
  - We don't always know the values of variables
  - Probabilities let us reason about variables even when we are uncertain

Engineering rationale behind probabilities

- Encoding uncertainty
  - Data are variables
  - We don't always know the values of variables
  - Probabilities let us reason about variables even when we are uncertain
- Encoding confidence
  - The flip side of uncertainty
  - Useful for decision making: should we trust our conclusion?
  - We can construct probabilistic models to boost our confidence
    - E.g., combining polls

Random variable

- Random variables take on values in a *sample space*.
- They can be *discrete* or *continuous*:
  - Coin flip: $\{H, T\}$
  - Height: positive real values $(0, \infty)$
  - Temperature: real values $(-\infty, \infty)$
  - Number of words in a document: Positive integers $\{1, 2, \ldots\}$
- We call the outcomes *events*.
- Denote the random variable with a capital letter; denote a realization of the random variable with a lower case letter.
  - E.g., X is a coin flip, x is the value (H or T) of that coin flip.

Discrete distribution

- A discrete distribution assigns a probability to every event in the sample space
- For example, if X is a coin, then

$$
\begin{aligned}
P(X=H) &= 0.5 \\
P(X=T) &= 0.5
\end{aligned}
$$

- And probabilities have to be greater than or equal to 0
- The probabilities over the entire space must sum to one

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$$\sum P(X = x) = 1$$

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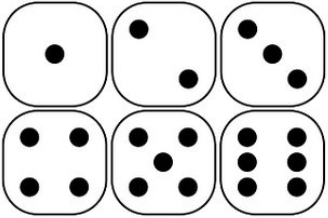
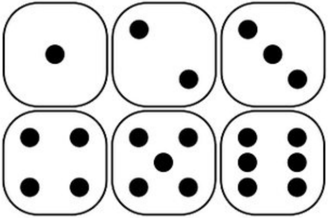
$$\sum_x P(X = x) = 1$$

A Fair Die



1	2	3	4	5	6
$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

A Fair Die



1	2	3	4	5	6
$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

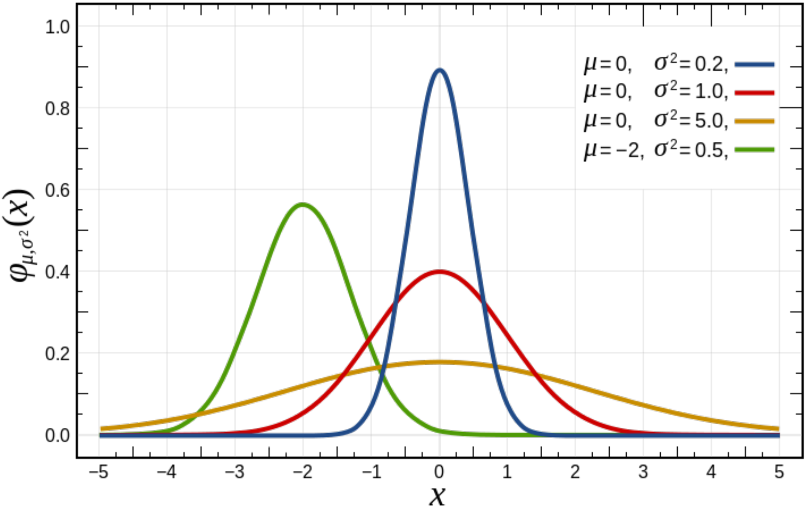
```python
def die_prob(x):
  if x in [0, 1, 2, 3, 4, 5, 6]:
    return 1.0 / 6.0
  else:
    return 0.0
```

The normal distribution

- The most common continuous distribution is the normal distribution, also called the Gaussian distribution.
- The density is defined by two parameters:
  - μ: the mean of the distribution
  - σ^2: the variance of the distribution (σ is the standard deviation)
- The normal density has a “bell curve” shape and naturally occurs in many problems.

Carl Friedrich Gauss
1777 – 1855

The normal distribution



The normal distribution

- The probability density of the normal distribution is:

$$f(x) = \underbrace{\frac{1}{\sqrt{2\pi\sigma^2}}}_{\substack{\text{Does not} \\ \text{depend on } x}} \underbrace{\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)}_{\substack{\text{Largest when } x=\mu; \\ \text{shrinks as } x \text{ moves} \\ \text{away from } \mu}}$$

- Notation: $\exp(x) = e^x$
- If X follows a normal distribution, then $\mathbb{E}[X] = \mu$.
- The normal distribution is symmetric around μ.

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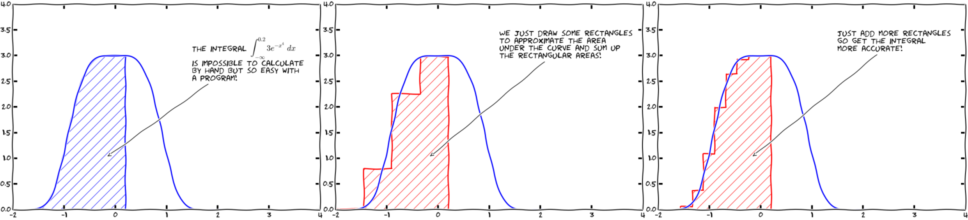

From Svein Linge and Hans Petter Langtangen

The normal distribution

- What is the probability that a value sampled from a normal distribution will be within *n* standard deviations from the mean?
- $P(\mu - n\sigma \leq X \leq \mu + n\sigma) = ?$

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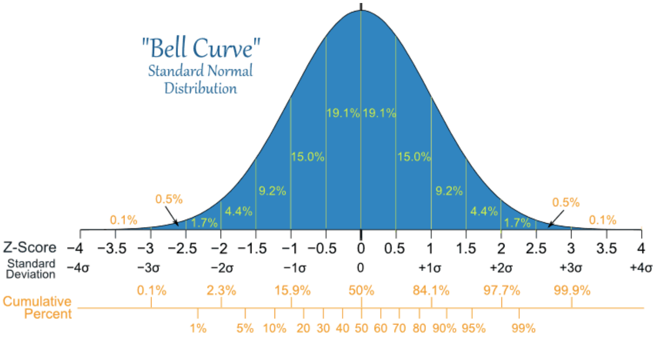
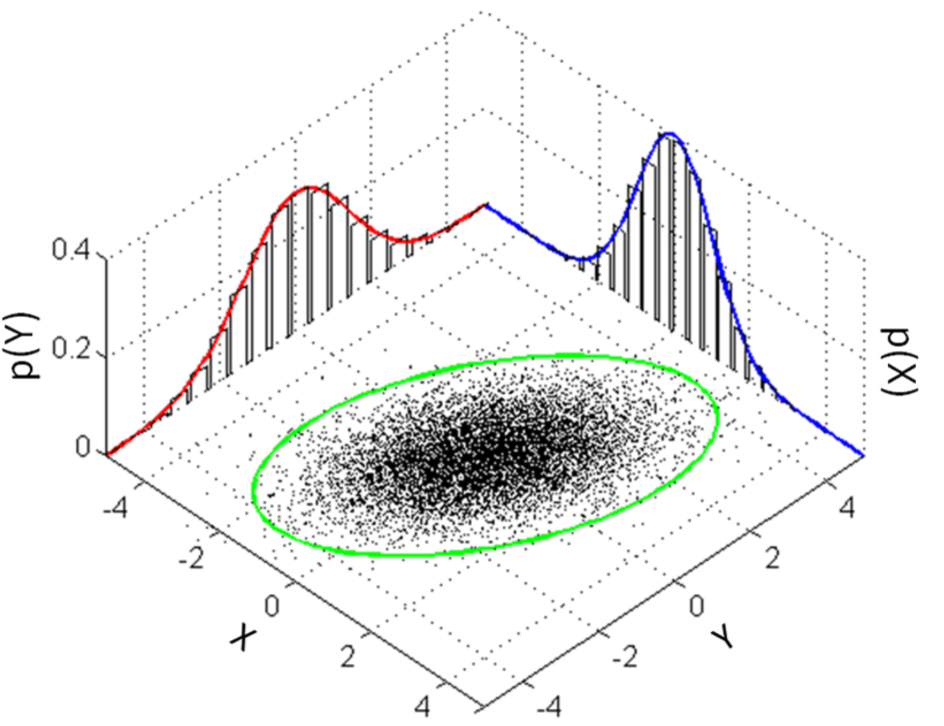
```
>>> from scipy.stats import norm
>>> norm.cdf(1.0) - norm.cdf(-1.0)
0.6826894921370859
```

The normal distribution





p(Y)
p(X)
0.4
0.2
0
-4
-2
0
2
4
X
4
2
0
-2
-4
Y

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Vectors

Row Vector

$$\vec{v} = \begin{bmatrix} 5 & 8 \end{bmatrix} \tag{21}$$

Column Vector

$$\begin{bmatrix} 5 \\ 2 \end{bmatrix} \tag{22}$$

Indexing elements

$v_1 = 5$; $v_2 = 8$

Vector Addition

$$\begin{bmatrix} 5 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 5+3 \\ 2+7 \end{bmatrix} = \begin{bmatrix} 8 \\ 9 \end{bmatrix} \tag{23}$$

Scalar Multiplication

$$3 \cdot \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \cdot 5 \\ 3 \cdot 2 \end{bmatrix} = \begin{bmatrix} 15 \\ 6 \end{bmatrix} \tag{24}$$

Dot Product Example

$$\begin{bmatrix} 4 \\ 3 \end{bmatrix}^\top \cdot \begin{bmatrix} 5 \\ 2 \end{bmatrix} =$$

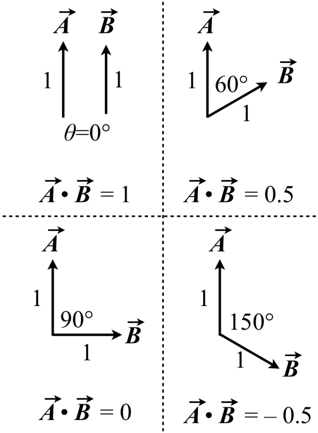
Dot Product Example

$$\begin{bmatrix} 4 \\ 3 \end{bmatrix}^\top \cdot \begin{bmatrix} 5 \\ 2 \end{bmatrix} = 4 \cdot 5 + 3 \cdot 2 = 26 \tag{25}$$

Dot Product Definition

$$\vec{x} \cdot \vec{y} = \sum_{i}^{D} x_i y_i \qquad (26)$$

$$\vec{x} \cdot \vec{y} = |\vec{x}||\vec{y}| \cos \theta \qquad (27)$$



From Scott Hill

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Dot Product Example

$$\begin{bmatrix}4\\3\end{bmatrix}^\top \cdot \begin{bmatrix}5\\2\end{bmatrix} = \tag{28}$$

$$\begin{bmatrix}4 & 3\end{bmatrix} \cdot \begin{bmatrix}5\\2\end{bmatrix} = \tag{29}$$

$$\begin{bmatrix}4 \cdot 5 + 2 \cdot 3\end{bmatrix} = \begin{bmatrix}26\end{bmatrix} \tag{30}$$

Dot Product Example

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$$\begin{bmatrix} 26 \end{bmatrix} \tag{30}$$

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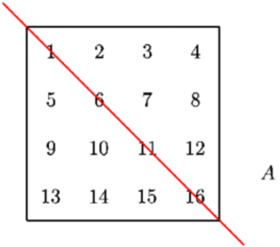
Transpose

- Turns *n* by *m* matrix into *m* by *n* matrix
- Swaps element in a_{ij} with element in a_{ji}



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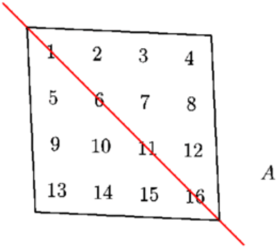

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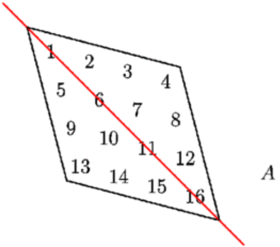

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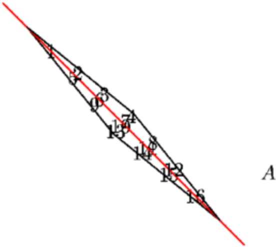

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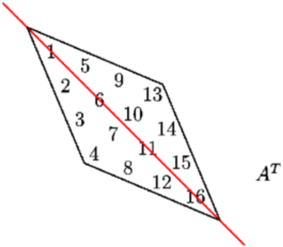

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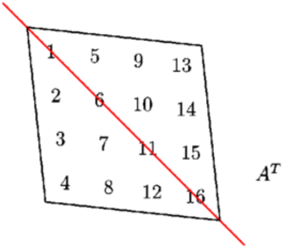

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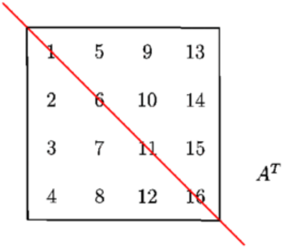
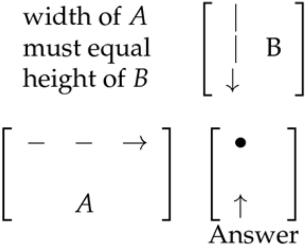


From Michael Doob

Matrix Multiplication Rules

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} \cdot & 0 \\ \cdot & 3 \\ \cdot & 0 \\ \cdot & 2 \end{bmatrix} = \begin{bmatrix} \cdot & 14 \\ \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}$$



From Denis Auroux

Matrix Multiplication with Identity

General Formula

$$a_{ij} = \sum_k l_{ik} r_{kj} \tag{31}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} \end{bmatrix} \tag{32}$$

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$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} ? \\ \end{bmatrix} \tag{32}$$

$a_{11} = l_{11} r_{11} + l_{12} r_{21} = 3 + 0 = 3$

Matrix Multiplication with Identity

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$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ ? \end{bmatrix} \tag{32}$$

$a_{21} = l_{21} r_{11} + l_{22} r_{21} = 0 + 4 = 4$

Matrix Multiplication with Identity

General Formula

$$a_{ij} = \sum_k l_{ik} r_{kj} \tag{31}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \tag{32}$$

Selecting a Row

$$\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}^\top \cdot \begin{bmatrix} 9 & 7 \\ 0 & 8 \\ 6 & 7 \\ 5 & 3 \\ 0 & 9 \end{bmatrix} = \begin{bmatrix} ? & \end{bmatrix} \tag{33}$$

Selecting a Row

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$$\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}^\top \cdot \begin{bmatrix} 9 & 7 \\ 0 & 8 \\ 6 & 7 \\ 5 & 3 \\ 0 & 9 \end{bmatrix} = \begin{bmatrix} 6 & 7 \end{bmatrix} \tag{33}$$

Selecting Rows

$$\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}^\top \cdot \begin{bmatrix} 9 & 7 \\ 0 & 8 \\ 6 & 7 \\ 5 & 3 \\ 0 & 9 \end{bmatrix} = \begin{bmatrix} ? & \quad \end{bmatrix} \tag{34}$$

Selecting Rows

$$\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}^\top \cdot \begin{bmatrix} 9 & 7 \\ 0 & 8 \\ 6 & 7 \\ 5 & 3 \\ 0 & 9 \end{bmatrix} = \begin{bmatrix} ? & ? \end{bmatrix} \tag{34}$$

Selecting Rows

$$\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}^\top \cdot \begin{bmatrix} 9 & 7 \\ 0 & 8 \\ 6 & 7 \\ 5 & 3 \\ 0 & 9 \end{bmatrix} = \begin{bmatrix} & ? \end{bmatrix} \tag{34}$$

Selecting Rows

$$\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}^\top \cdot \begin{bmatrix} 9 & 7 \\ 0 & 8 \\ 6 & 7 \\ 5 & 3 \\ 0 & 9 \end{bmatrix} = \begin{bmatrix} 9+0+0 & ? \end{bmatrix} \tag{34}$$

Selecting Rows

$$\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}^\top \cdot \begin{bmatrix} 9 & 7 \\ 0 & 8 \\ 6 & 7 \\ 5 & 3 \\ 0 & 9 \end{bmatrix} = \begin{bmatrix} 9 & ? \end{bmatrix} \tag{34}$$

Selecting Rows

$$\begin{bmatrix}1\\1\\0\\0\\1\end{bmatrix}^\top \cdot \begin{bmatrix}9 & 7\\0 & 8\\6 & 7\\5 & 3\\0 & 9\end{bmatrix} = \begin{bmatrix}9 & 7+8+9\end{bmatrix} \tag{34}$$

Selecting Rows

$$\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}^{\top} \cdot \begin{bmatrix} 9 & 7 \\ 0 & 8 \\ 6 & 7 \\ 5 & 3 \\ 0 & 9 \end{bmatrix} = \begin{bmatrix} 9 & 24 \end{bmatrix} \tag{34}$$