Faculty of Engineering
EM 509 – Stochastic Processes
Answer Sheet
Classification

1) Consider the random process, $X(t) = A\cos(2\pi t)$, where A is a random variable:
Let the mean and variance of A are μ_A and σ_A^2 respectively.

$$\text{Mean} = \mu_x(t) = E(A\cos(2\pi t))$$
$$= E(A)E(\cos(2\pi t))$$
$$= E(A)\cos(2\pi t)$$
$$= \mu_A \cos(2\pi t)$$

$$\text{Variance} = \sigma_X^2(t) = E(X_t - E(X_t))^2$$
$$= E(A\cos(2\pi t) - E(A)\cos(2\pi t))^2$$
$$= E(\cos(2\pi t)(A - E(A))^2$$
$$= E(\cos^2(2\pi t))E(A - E(A))^2$$
$$= \cos^2(2\pi t)\sigma_A^2$$

$$\text{Autocorrelation} = R_X(t_1, t_2) = E(A\cos(2\pi t_1)A\cos(2\pi t_2))$$
$$= E(A^2)\cos(2\pi t_1)\cos(2\pi t_2)$$

$$\text{Autocovariance} = \mathrm{R}_{\mathrm{X}}(t_1, t_2) - \mu_X(t_1)\mu_X(t_2)$$
$$= E(A^2)\cos(2\pi t_1)\cos(2\pi t_2) - E(A)\cos(2\pi t_1)E(A)\cos(2\pi t_2)$$
$$= (E(A^2) - (E(A))^2)\cos(2\pi t_1)\cos(2\pi t_2)$$
$$= \sigma_A^2 \cos(2\pi t_1)\cos(2\pi t_2)$$

2) Consider the random process $Y_n = 2X_n - 1$, where X_n is a Bernoulli process with the probability of success and failure respectively given by $P(X_n = 1) = p$ and $P(X_n = 0) = 1 - p$.
The possible outcomes for Y_n,

$$Y_n = \begin{cases} 1 \ ; X_n = 1 \\ -1; X_n = 0 \end{cases}$$

$$\text{Mean} = \mu_Y(n) = E(Y_n)$$
$$= E(2X_n - 1)$$
$$= 2E(X_n) - 1$$
$$= 2p - 1$$

$$\text{Variance} = \sigma_Y^2 = E(Y_n - E(Y_n))^2$$
$$= E(2X_n - 1 - E(2X_n - 1))^2$$
$$= E(2X_n - 1 - 2E(X_n) + 1)^2$$
$$= E(2X_n - 2E(X_n))^2$$
$$= 4E(X_n - E(X_n))^2$$
$$= 4p(1-p)$$

$$\text{Autocorrelation} = R_Y(n,m) = E(Y_n Y_m)$$
$$= E((2X_n - 1)(2X_m - 1))$$
$$= E(4X_n X_m - 2X_n - 2X_m + 1)$$
$$= 4E(X_n X_m) - 2E(X_n) - 2E(X_m) + 1$$
$$= 4p^2 - 4p + 1 = (2p-1)^2$$

3) The random process $Y_n = 2X_n - 1$, where X_n is a Bernoulli process with the probability of success and failure respectively given by $P(X_n = 1) = p$ and $P(X_n = 0) = 1 - p$. N ow consider the one dimensional random work, $Z_n = Y_1 + Y_2 + ... + Y_n ; n = 1,2,3,...$

$$\text{Mean} = \mu_Z(n) = E(Z_n) = E(\sum_{i=1}^{n} Y_i)$$
$$= \sum_{i=1}^{n} E(Y_i)$$
$$= n(2p-1)$$

$$\text{Variance} = \sigma_Z^2 = E((Z_n)^2) - (E(Z_n)^2)$$
$$= E((Z_n)^2) - n^2(2p-1)^2$$

Consider;

$$E((Z_n)^2) = E(\sum_{i=1}^{n} Y_i \sum_{j=1}^{n} Y_j) = E(\sum_{i=1}^{n} \sum_{j=1}^{n} Y_i Y_j) = \sum_{i=1}^{n}\sum_{j=1}^{n} E(Y_i Y_j)$$
$$= \sum_{\substack{i=1 \\ i\neq j}}^{n}\sum_{j=1}^{n} E(Y_i Y_j) + \sum_{i=1}^{n} E((Y_i)^2)$$
$$= \sum_{\substack{i=1 \\ i\neq j}}^{n}\sum_{j=1}^{n} E(Y_i)E(Y_j) + \sum_{i=1}^{n} 1$$
$$= (n^2 - n)(2p-1)^2 + n$$

Note that: $P(Y_n = 1) = p$ and $P(Y_n = -1) = 1 - p$ as $P(X_n = 1) = p$ and $P(X_n = 0) = 1 - p$

Then, $E(Y_i^2) = (1^2 p + (-1)^2 (1 - p)) = 1$

$E(Y_i Y_j) = E(Y_i)E(Y_j) = \mu_Y^2 = (2p - 1)^2$

$$\begin{aligned}
\text{Variance} = \sigma_Z^2 &= E((Z_n)^2) - n^2(2p-1)^2 \\
&= (n^2 - n)(2p-1)^2 + n - n^2(2p-1)^2 \\
&= n - 4np^2 + 4np - n \\
&= 4np(1-p)
\end{aligned}$$

$$\begin{aligned}
\text{Autocorrelation} = R_Z(n,m) &= E(Z_n Z_m) \\
&= E(Z_n(Z_n + \sum_{j=n+1}^{m} Y_j) \\
&= E(Z_n^2) + E(Z_n \sum_{j=n+1}^{m} Y_j) \\
&= E(Z_n^2) + E(\sum_{i=1}^{n} Y_i \sum_{j=n+1}^{m} Y_j) \\
&= E(Z_n^2) + \sum_{i=1}^{n} \sum_{j=n+1}^{m} E(Y_i Y_j) \\
&= E(Z_n^2) + \sum_{i=1}^{n} \sum_{j=n+1}^{m} E(Y_i)E(Y_j) \\
&= E(Z_n^2) + n(m-n)(2p-1)^2; \text{Suppose that } m > n \\
&= E(Z_m^2) + m(n-m)(2p-1)^2; \text{Suppose that } m < n \\
&= (n^2 - n)(2p-1)^2 + n + n(m-n)(2p-1)^2; \text{Suppose that } m > n \\
&= (m^2 - m)(2p-1)^2 + m(n-m)(2p-1)^2; \text{Suppose that } m < n \\
&= nm(2p-1)^2 + \min(n,m)(1 - (2p-1)^2); \text{the genaralized equation}
\end{aligned}$$

4) Let X_t be a random process with mean function $\mu_X(t)$. Suppose that X_t is applied to a linear time-invariant (LTI) system with impulse response *h(t).*

$$Y_t = \int_{-\infty}^{\infty} h(t-\theta) X_\theta d\theta$$

Now consider,

$$\mu_Y(t) = E(Y_t)$$

$$= E\left[\int_{-\infty}^{\infty} h(t-\theta)X_\theta d\theta\right]$$

$$\approx E\left[\sum_i h(t-\theta_i)X_{\theta_i}\Delta\theta_i\right]$$; Note that the integalion by writing the integral as a Rieman sum

$$= \sum_i E\left[h(t-\theta_i)X_{\theta_i}\Delta\theta_i\right]$$

$$= \sum_i E\left[h(t-\theta_i)X_{\theta_i}\right]\Delta\theta_i$$

$$\approx \int_{-\infty}^{\infty} E\left[h(t-\theta)X_\theta\right]d\theta$$

$$= \int_{-\infty}^{\infty} h(t-\theta)E(\left[X_\theta\right])d\theta;$$

Since $h(t-\theta)$ is a non - random variable and can be pulled out of the expectation

$$= \int_{-\infty}^{\infty} h(t-\theta)\mu_X(\theta)d\theta$$