On the Border Between Recreational and "Serious" Mathematics: Rectangle Free Coloring of Grids

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Abstract

LATER

1 Introduction

Let K be a field and let $r \in K$. We will consider polynomials of the form

$$f_r(x) = x^2 + r.$$

We will often suppress the subscript of r.

We will ask questions about *iterations* of f.

2 Number of Factors

Look at $f(x) = x^2$.

Iteration	Factors	Number of Factors
$f^1_(x) = f(x) = x^2$	$x \times x$	2
$f^2(x) = f(f(x)) = x^4$	$x \times x \times x \times x$	4

This is easy! $f^i(x)$ has 2^i factors.

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Lets try something harder. $f(x) = x^2 - 1$.

1. $f^1(x) = x^2 - 1 = (x+1)(x-1)$.

   2 factors.

2. $f_2(x) = (x^2-1)^2 - 1 = (x^2-2)x^2$.

   3 factors.

3. $f^3(x) = (x^2-2)^2x^4 - 1 = (x^4 - 2x^2 - 1)(x-1)^2(x+1)^2$

   factors

4. $f^4(x) = (x^4 - 2x^2 - 1)^2(x-1)^4(x+1)^4 - 1$

   $(x^7 - x^6 - 3x^5 + 3x^4 + x^3 - x^2 + x - 2)x(x^6 - x^5 - 3x^4 + 3x^3 + x^2 - x + 1)$

   3 factors.

5. $f^5(x) =$

   $(x^7 - x^6 - 3x^5 + 3x^4 + x^3 - x^2 + x - 2)^2x^2(x^6 - x^5 - 3x^4 + 3x^3 + x^2 - x + 1)^2 - 1 =$

   $(x^{14} - 2x^13 - 5x^12 + 12x^11 + 5x^10 - 22x^9 + 7x^8 + 8x^7 - 9x^6 + 10x^5 - 3x^4 - 4x^3 + 3x^2 - 2x - 1) \times$
   $(x-1)^4(x+1)^2(x^8 - 4x^6 + 2x^4 + 4x^2 + 1)$

   8 factors

The following is known.

Theorem 2.1 *Let $a(n)$ be the number of factors of $f^n(x)$ that are irreducible in $\mathsf{Z}[x]$. Then*

$$a(n) = \begin{cases} 2 \times 2^{n/2} - 1 & \text{if } n \text{ is even;} \\ 3 \times 2^{(n-1)/2} - 1 & \text{if } n \text{ is odd.} \end{cases} \tag{1}$$

SEE IF THERE IS AN ELEMENTARY PROOF