ON SMALL GRAPHS WITH FORCED MONOCHROMATIC TRIANGLES

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Let us denote by $S(k,\ell;\ r)$ the following statement:

There exists a graph G which does not contain a complete subgraph on ℓ vertices but which has the property that any r-coloring of the edges of G must contain a monochromatic complete subgraph on k vertices.

It is immediate from Ramsey's Theorem (cf. [5]) that for any fixed k and r, $S(k,\ell;\ r)$ is true for ℓ sufficiently large. In partiular, it follows that $S(3,7;\ 2)$ holds by taking G to be K_6, the complete graph on 6 vertices. Recently, Erdös and Hajnal [1] asked whether $S(3,6;\ 2)$ holds. This was first answered affirmatively by J. H. van Lint (unpublished) who gave as an example of a graph which establishes $S(3,6;\ 2)$, the <u>complement</u> of the graph shown in Fig. 1.

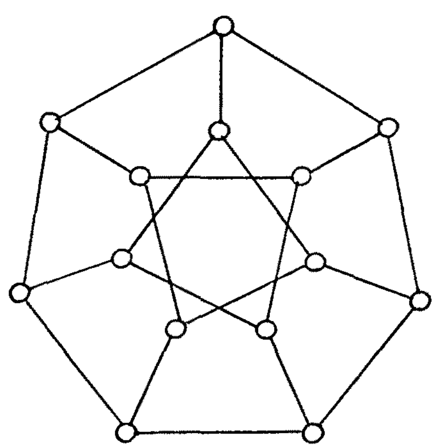


Figure 1

Soon thereafter, L. Pósa (unpublished) proved the existence of a graph G for which $S(3,5;\ 2)$ holds, basing his work on some previous existence proofs of Erdös.

The final step in this direction was achieved by the late J. H. Folkman [2] who established $S(3,4;\ 2)$ by the explicit construction of an appropriate (very large) graph G. More generally, Folkman also established $S(k,k+1;\ 2)$ in [2] for all $k \geq 3$. Furthermore, Folkman asserted in 1968 that he had a proof of $S(3,4;\ 3)$ and a very complicated proof of $S(3,4;\ 4)$ but no notes on these ideas have as of yet been discovered. It was conjectured by Folkman and independently by Erdös and Hajnal that $S(k,k+1;\ r)$ holds for all k and r.

Erdös has pointed out that it would be of interest to determine the least number $N(k,\ell;\ r)$ of vertices a graph may have which can be used to establish $S(k,\ell;\ r)$. It was shown by one of the authors in [3] that $N(3,6;\ 2) = 8$. The unique graph G which achieves this bound is the complement of the 8 vertex graph shown in Fig. 2. Thus, G has 8 vertices and 23 edges.

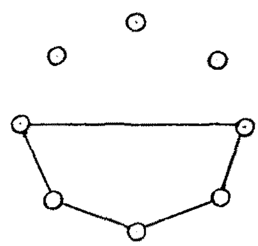


Figure 2

The results of [2] show that $N(3,4;\ 2) < \infty$. In a recent paper, Schäuble [6] proves $N(3,5;\ 2) \leq 42$ by considering the graph shown in Fig. 3.

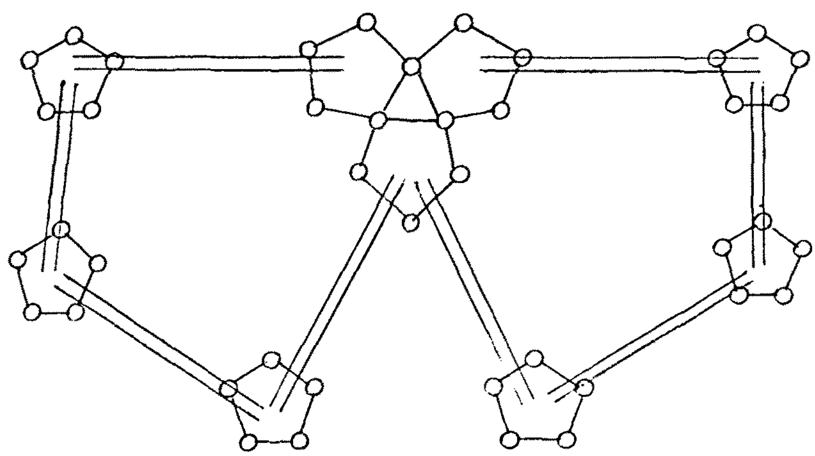
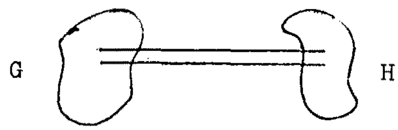


Figure 3

Here, we use the notation



to indicate that all vertices of G are connected to all vertices of H.

In this note we prove the following result:

<u>Theorem</u>: $N(3,5;\ 2) \leq 23$.

<u>Proof</u>: Consider the graph G given in Fig. 4.

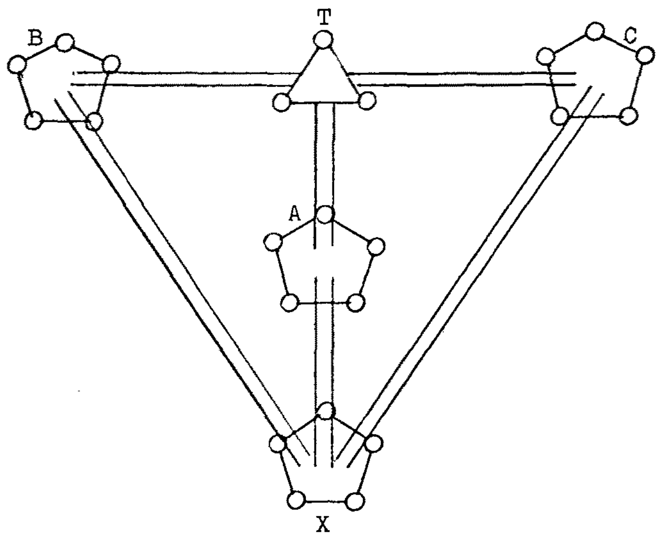


Figure 4

In G, each vertex of pentagon A is just connected to the vertices t_2 and t_3 of triangle T, each vertex of B is connected to vertices t_1 and t_2 of T, and each vertex of C is connected to vertices t_1 and t_3. All vertices of pentagon X are connected to all vertices of pentagons A, B, C. Thus, G has 23 vertices and 128 edges. We must show that G can be used to establish $S(3,5;\ 2)$.

(i) <u>$K_5 \not\subseteq G$</u>. Consider the possible locations of the vertices of a hypothetical subgraph K_5. We cannot have ≥ 3 vertices of this K_5 in one pentagon A, B, C or X since they all contain no triangles. Also, since there are no edges between pentagons A, B and C, no vertex of the K_5 can be in X. If the K_5 had ≥ 3 vertices not in T, at least <u>two</u> of the pentagons A, B, C would have to contain a vertex of the K_5

which is impossible since these pentagons have no interconnecting edges. The only possibility left is if all 3 vertices of T were also vertices of the K_5. The remaining 2 vertices of the K_5 must then belong to one of A, B, C which is also impossible.

(ii) Any 2-coloring of the edges of G contains a monochromatic triangle. We need two preliminary facts to establish (ii). We refer to Fig. 5 for the graphs under consideration. Assume the graphs H_1 and H_2 have been 2-colored so that no monochromatic triangles have been formed.

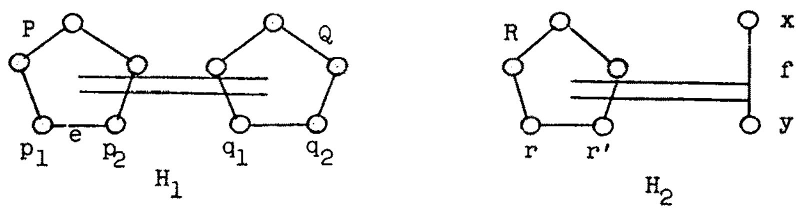


Figure 5

(a) All edges of the pentagons P and Q of H_1 must be the same color. This fact was used by Schauble in [6]. We indicate a short proof. Assume some edge e of P is red. If ≥ 3 of the edges from some endpoint p_1 of e to Q were red then 2 of these edges must go to adjacent vertices of Q, say, q_1 and q_2. But if any edge between p_2, q_1, q_2 is red then we get a red triangle; if they are all blue then we get a blue triangle. Thus, at most 2 of the edges from p_1 to Q can be red, i.e., at least 3 of them are blue. Of course, this is also true for the other endpoint of e. But this implies that any edge of P adjacent to e must also be red since they share a common endpoint. Hence, all edges of P are red. Hence, at least 3/5 of all the edges between P and Q must be blue which implies by symmetry that all the edges of Q are also red. This proves (a).

(b) If all edges of pentagon R of H_2 are red then the edge f is red. Assume f is blue. For each vertex r of R consider the ordered pair of colors $(C_x(r), C_y(r))$ where $C_x(r)$ is the color assigned to the edge from r to x, with $C_y(r)$ defined similarly. We certainly cannot have $(C_x(r), C_y(r))$ = (blue,blue) since this forms a blue triangle r,x,y. Also $(C_x(r), C_y(r))$ = (red,red) is impossible because any red edge between r',x,y forms a red triangle and if these edges are all blue then a blue triangle is formed. Hence, we must have $(C_x(r), C_y(r))$ = (red,blue) or (blue,red). However, we cannot have $(C_x(r), C_y(r)) = (C_x(r'), C_y(r'))$ because the red component, say, $C_x(r) = C_x(r')$ = red, would form a red triangle r,r′,x. Hence adjacent vertices in H_2 must have distinct pairs $(C_x(r), C_y(r))$. This is impossible however because H_2 is an odd cycle. This proves (b).

The proof of (ii) is now immediate. Assume without loss of generality that some edge of pentagon X in G is red. Hence by (a), all edges of A, B and C are also red. Finally, by (b), all edges of triangle T are red. This proves the Theorem.

It might be conjectured that $N(3,5;\ 2) = 23$ although admittedly there is not too much evidence for such an assertion. It seems very difficult to establish any nontrivial lower bounds on the $N(k,\ell;\ r)$. S. Lin [4] has recently shown $N(3,5;\ 2) \geq 10$. However, it is not known even if $N(3,5;\ 2) \geq 11$.

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