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Large Triangle-Free Subgraphs in Graphs without K_4

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Abstract. It is shown that for arbitrary positive ε there exists a graph without K_4 and so that all its subgraphs containing more than $1/2 + \varepsilon$ portion of its edges contain a triangle (Theorem 2). This solves a problem of Erdös and Nešetřil. On the other hand it is proved that such graphs have necessarily low edge density (Theorem 4).

Theorem 3 which is needed for the proof of Theorem 2 is an analog of Goodman's theorem [8], it shows that random graphs behave in some respect as sparse complete graphs.

Theorem 5 shows the existence of a graph on less than 10^{12} vertices, without K_4 and which is edge-Ramsey for triangles.

1. Introduction

Let $G = (V, E)$ be a simple graph without loops or multiple edges (for notions from graph theory we refer to [2]). An averaging argument shows that for any $k \geq 2$ there is a k-chromatic complete (complete k-partite) graph $H = (V, E')$ so that $|E \cap E'| > \left(1 - \frac{1}{k}\right)|E|$ holds (cf. [2]). Since H contains no K_{k+1}, we have the following.

Proposition 1. *For every $k \geq 2$, every graph with e edges contains a subgraph without K_{k+1} with more than $\left(1 - \frac{1}{k}\right) e$ edges.* □

In the case of $k = 2$ we obtain a triangle-free subgraph with more than half of the edges. Erdös and Nešetřil [5] asked whether the constant 1/2 can be improved if we make the additional assumption: G is K_4-free. We answer this question in the negative by proving: ($|G|$ denotes the number of edges of G).

Theorem 2. *For an arbitrary positive ε there exists a K_4-free graph G so that all its subgraphs G_0 with $|G_0| \geq (\frac{1}{2} + \varepsilon)|G|$ contain a triangle.*

For the proof we need a result of independent interest ($t(G)$ is the number of triangles in G).

Theorem 3. *Suppose $R = R(n, p)$ is the probability space of all random graphs on n* *vertices with edge probability p*, $1 > p > n^{\varepsilon - 1/2}$, *for some positive constant* ε. *Let* γ *be a constant*, $0 < \gamma < 1$ *and suppose R is partitioned into edge-disjoint subgraphs* R_1, R_2 *with* $|R_1| \sim \gamma |R|$. *Then for almost all* $\bar{R} \in R(n, p)$

$$t(R_1) + t(R_2) \gtrsim \frac{1 + 3(1 - 2\gamma)^2}{4} t(\bar{R})$$

holds for all partitions $\bar{R} = R_1 \cup R_2$ *with the above property.* (We say that a statement holds for almost all $\bar{R} \in R$ if it holds with probability $1 - o(1)$, where $o(1) \to 0$ as $n \to \infty$.)

Remark. The corresponding statement for the complete graph was proved by Goodman [8]. Thus our theorem shows that random graphs behave as complete graphs, i.e. they are like sparse complete graphs. Note that our notation $x \gtrsim y$ means that for any $\delta > 0$ we have $x \geq (1 - \delta) y$ for $n > n_0(\delta, \varepsilon)$. Note also that the constant $(1 + 3(1 - 2\gamma)^2)/4$ is easily seen to be best possible.

The graph G constructed in Theorem 2 is sparse – it has $m^{3/2+\varepsilon}$ edges $(0 < \varepsilon < 0.1$ and m is the number of vertices). Replacing each vertex v of G by $\left\lceil \frac{n}{m} \right\rceil = t$ other vertices $v_1, v_2, \ldots, v_t$ and joining v_i, v_j', $1 \leq i, j \leq t$, if and only if v and v' are joined in G we get a new graph which shows that for every fixed $\varepsilon > 0$ there exists a positive constant c_ε (independent of n) and graphs having n vertices and $c_\varepsilon n^2$ edges and still having the property of the graph from Theorem 1.2. Let $\bar{c}_\varepsilon$ be the supremum of all c_ε's with the above property. P. Erdös [5] conjectured that $\lim_{\varepsilon \to 0} \bar{c}_\varepsilon = 0$. We prove this here and show the following slightly stronger statement.

Theorem 4. *Let a positive integer k and a positive real c*, $0 < c < 1$ *be given. Then there exists* $n_0 = n_0(k, c)$ *and* $\varepsilon = \varepsilon(k, c)$ *such that the following holds: If G is a graph on n vertices and with* $c\binom{n}{2}$ *edges* $n > n_0$ *which has the property that every bipartite subgraph of it has less then* $\left(\frac{1}{2} + \varepsilon\right) c \binom{n}{2}$ *edges, then G contains* K_k.

P. Erdös and A. Hajnal conjectured that for every k there exists a graph G_k which contains no K_{k+1} but if one colors the edges of G_k by two (or in general p) colors in an arbitrary way there is always a monochromatic K_k. Folkman [7] proved this conjecture for $p = 2$ and the general conjecture was settled by Nešetřil and Rödl [12] – in fact they proved a more general theorem. However, there are many numerical problems which remain. Let $f(p, k_1, k_2)$ be the smallest integer n for which there is a graph G with n vertices not containing K_{k_2} but if we color the edges of g by p colors there is always a monochromatic K_{k_1}. Graham [9] proved that $f(2, 3, 6) = 8$ and Irving [11] proved $f(2, 3, 5) \leq 18$. On the other hand both Folkman's and Nešetřil, Rödl's upper bounds for $f(2, 3, 4)$ are extremely large (greater than ten times iterated exponential). P. Erdös [4] offered max {100 dollars, 300 swiss francs} for a proof or disproof of $f(2, 3, 4) < 10^{10}$. Unfortunately, we were not able to settle this problem. However, the method of proof of Theorem 2 allows to show the following.

Theorem 5. $f(2,3,4) < 7.02 \cdot 10^{11}$.

It was pointed out by Noga Alon [1] that this result either proves or disproves a conjecture of R.L. Graham [10, p. 36].

In the proofs we will assume several elementary properties of random graphs (e.g. (3)). Since each of these properties holds with probability tending to 1, the same holds for any finite number of them. For non-proved statements concerning random graphs we refer to [6]. Note also that we shall often identify graphs with their edge sets.

2. The proof of Theorem 3

Let $\{1, 2, \ldots, n\}$ be the vertex set of R. Since the edges are chosen independently each with probability p, we have

$$\mathrm{Prob}(\{i,j,k\} \text{ is a triangle}) = p^3. \tag{1}$$

Consequently with probability tending to one we have

$$t(R) \sim \binom{n}{3} p^3. \tag{2}$$

Similarly, the degree d_i of vertex i satisfies

$$d_i \sim np \quad \text{for every} \quad i = 1, 2, \ldots, n. \tag{3}$$

Moreover, if X_i, $|X_i| = x_i$ is a subset of the neighborhood of the i-th vertex, one can infer that

$$\left| |R \cap [X_i]^2| - \binom{x_i}{2} p \right| = o(n^2 p^3) \tag{4}$$

holds again with probability $1 - o(1)$ simultanously for all vertices and all X_i. Let $\bar{R}$ be any graph having properties (2), (3) and (4) (almost all graphs have these properties) and let $\bar{R} = R_1 \cup R_2$ be a partition of the edges of $\bar{R}$. Let us call the edges in R_1 blue, those in R_2 red. Denote by x_i the blue degree of the vertex i. It follows from (3) and (4) that

$$|N_{\bar{R}}(i)| \sim \binom{d_i}{2} p \sim \frac{n^2 p^3}{2} \tag{5}$$

$$\left| |N_{R_1}(i)| - \binom{x_i}{2} p \right| = o(n^2 p^3) \tag{6}$$

$$\left| |N_{R_2}(i)| - \binom{d_i - x_i}{2} p \right| = o(n^2 p^3) \tag{7}$$

where $N_{\bar{R}}(i)$ denotes the edge set of the neighborhood of the vertex i in the graph $\bar{R}$ ($N_{R_1}(i)$ and $N_{R_2}(i)$ are defined analogously) and d_i is the size of the neighborhood (vertex set) of vertex i.

Let $t_b(t_r)$ denote the number of non-monochromatic triangles with 2 blue (2 red) edges, respectively. Each edge in $N_{R_1}(i)$ gives rise to a blue or to a non-monochromatic triangle with two blue edges adjacent to i. Summing up over i we infer using (6)

$$3t(R_1) + t_b = \sum_{i=1}^{n} \frac{x_i^2 p}{2} + o(n^3 p^3). \tag{8}$$

The same consideration for $N_{R_2}(i)$ gives

$$3t(R_2) + t_r = \sum_{i=1}^{n} \frac{(d_i - x_i)^2 p}{2} + o(n^3 p^3). \tag{9}$$

From (5), (6) and (7) it follows that the number of edges in $N_{\bar{R}}(i) - (N_{R_1}(i) \cup N_{R_2}(i))$ is asymptotic to $x_i(d_i - x_i)p$. Since those edges give rise to non-monochromatic triangles, we infer

$$2(t_b + t_r) = \sum_i x_i(d_i - x_i)p + o(n^3 p^3). \tag{10}$$

From (8) + (9) − (10) we obtain:

$$3(t(R_1) + t(R_2)) - (t_b + t_r) = \frac{p}{2}\sum_i (d_i - 2x_i)^2 + o(n^3 p^3) \tag{11}$$

As $\sum_i (d_i - 2x_i) = 2|R| - 4|R_1| = (2 - 4\gamma)|R|$ holds, $\sum (d_i - 2x_i)^2$ is minimal if $d_i - 2x_i = \frac{2 - 4\gamma}{n}|R| = pn(1 - 2\gamma)$. Thus (11) yields in view of $t_b + t_r + t(R_1) + t(R_2) = t(R)$

$$4(t(R_1) + t(R_2)) \gtrsim t(R) + \frac{p^3 n^3}{2}(1 - 2\gamma)^2 \quad \text{or using (2)}$$

$$t(R_1) + t(R_2) \gtrsim \frac{1 + 3(1 - 2\gamma)^2}{4} t(R). \qquad \square$$

The Proof of Theorem 2. Let us consider a random graph $\bar{R} \in R(n, p)$, with edge probability $p = n^{\varepsilon - 1/2}$, $0 < \varepsilon < 0.1$. The expected number of K_4's in $\bar{R}$, $E(k_4(\bar{R}))$ satisfies $E(k_4(\bar{R})) = \binom{n}{4} p^6 \sim \frac{n}{24} n^{6\varepsilon}$ and thus almost all

$$\bar{R} \in R \text{ have at most } 2E(k_4(\bar{R})) \sim \frac{n}{12} n^{6\varepsilon}\ K_4\text{'s}. \tag{12}$$

For $\bar{R} \in R$ denote by $e(\bar{R})$ the set of edges in $\bar{R}$ which are contained in some K_4. It follows from (12) that

$$E(e(\bar{R})) \lesssim \frac{n^{1+6\varepsilon}}{2} \text{ holds} \tag{13}$$

with probability $1 - o(1)$.

$\bar{R} - e(\bar{R})$ is clearly a graph without K_4 and moreover

$$|R - e(R)| \sim \frac{1}{2} n^{3/2+\varepsilon} - \frac{1}{2} n^{1+6\varepsilon} \sim \frac{1}{2} n^{3/2+\varepsilon} \tag{14}$$

holds with probability $1 - o(1)$.

$\bar{R}$ has further the property that with probability $1 - o(1)$ each edge e is contained in

$$\sim np^2 \text{ triangles in } \bar{R} \text{ and there are } \sim \binom{n}{2} p \text{ edges in } \bar{R}. \tag{15}$$

Consider now $\bar{R} \in R$ from Theorem 3 having properties (12), (13), (14) and (15). We claim that $G = \bar{R} - e(\bar{R})$ is a good choice for Theorem 2. Suppose for contradiction $\varepsilon_0 > 0$ is given and R_1 is a triangle-free subgraph of G with

$$|R_1| > \left(\frac{1}{2} + \varepsilon_0\right)|G| \sim \left(\frac{1}{2} + \varepsilon_0\right)|\bar{R}|. \tag{16}$$

Set $R_2 = \bar{R} - R_1$ and apply Theorem 3 with $\gamma = \frac{1}{2} + \varepsilon_0$. Noting that R_1 is triangle-free we obtain

$$t(R_2) \gtrsim \left(\frac{1}{4} + 3\varepsilon_0^2\right) t(R).$$

Let us count the number of pairs, say m_i $(i = 1, 2)$ of the form (e, T), e is an edge of the triangle T, $e \in R_i$, T is in $\bar{R}$. It follows from (15) that

$$m_i \sim |R_i| np^2. \tag{17}$$

On the other hand each triangle in R_2 contributes 3 to m_2 and zero to m_1 while the remaining triangles in R contributes at least 1 to m_2 and at most 2 to m_1. Using (17) we infer

$$m_2 \gtrsim 3\left(\frac{1}{4} + 3\varepsilon_0^2\right)t(R) + \left(\frac{3}{4} - 3\varepsilon_0^2\right)t(R) = \left(\frac{3}{2} + 6\varepsilon_0^2\right)t(R)$$

$$m_1 \lesssim 2\left(\frac{3}{4} - 3\varepsilon_0^2\right)t(R) = \left(\frac{3}{2} - 6\varepsilon_0^2\right)t(R).$$

For $n > n_0(\varepsilon)$ using (17) this leads to $|R_2| > |R_1|$, contradicting (16). □

3. The Proof of Theorem 4

Before we give a proof we introduce (without proof) the following easy Lemma (see [14] for various generalizations).

Let $G = (V, E)$ be a graph, we define the density $d(G)$ of G by

$$d(G) = \frac{|E|}{|[V]^2|}.$$

Lemma. *Let $G_n = (V_n, E_n)$, $(|V_n| \to \infty)$ be a sequence of graphs with the property that whenever G_n^* is the subgraph of G_n having $\left\lfloor \frac{|V_n|}{2} \right\rfloor$ vertices then $\lim_{n\to\infty} d(G_n) = \lim_{n\to\infty} d(G_n^*) = c > 0$. Then for every k there exists n_k such that G_n contains a complete graph K_k for every $n \geq n_k$.*

$$\left.\begin{array}{l}\text{Let } G_n = (V_n, E_n) \text{ be a sequence of graphs not containing} \\ K_k, |E_n| = c_n\binom{v_n}{2}, \text{ having the property that} \\ \text{every bipartite subgraph of } G_n \text{ has less than} \\ \left(\frac{1}{2}+\varepsilon_n\right)c_n\binom{v_n}{2} \text{ edges where } \varepsilon_n \to 0, \text{ and } |V_n| \to \infty \text{ as } n \to \infty.\end{array}\right\} \quad (*)$$

$(|V_n| = v_n)$.

Suppose further that $c_n \to c > 0$ as $n \to \infty$ and c is as large as possible. Let, for every n, $G_n^* = (V_n^*, E_n^*)$ be a subgraph of G_n induced on $\left\lceil \frac{v_n}{2} \right\rceil$-subset of V_n and having as many edges as possible.

$$\left.\begin{array}{l}\text{Suppose that there exists } \varepsilon' > 0 \text{ and an infinite sequence} \\ \{G_{n_m}^*\}_{m=1}^{\infty} = \{H_m^*\} \text{ such that all of the graphs } H_m^* \text{ contain a} \\ \text{bipartite subgraph } H_m = (X_1^m \cup X_2^m, F_m) \text{ of } \left(\frac{1}{2}+\varepsilon'\right)|E_{n_m}^*| \text{ edges.}\end{array}\right\} \quad (**)$$

Set $W_m = V_{n_m} - V_{n_m}^*$, by a simple averaging argument (Proposition 1) we can find a bipartite subgraph $(Y_1^m \cup Y_2^m, F_m)$ of $(W_m, [W_m]^2 \cap E_{n_m})$ which has at least $\frac{1}{2}|E_{n_m} \cap [W_m]^2|$ edges. Thus the bipartite subgraph of G_{n_m} with bipartition either $(X_1^m \cup Y_1^m, X_2^m \cup Y_2^m)$ or $(X_1^m \cup Y_2^m, X_2^m \cup Y_1^m)$ has for m sufficiently large at least $\left(\frac{1}{2}+\frac{\varepsilon'}{4}\right)|E_{n_m}|$ edges – which contradicts our assumption on the sequence $\{G_n\}_{n=1}^{\infty}$. Thus (**) does not hold and all but finitely many members of the sequence $\{G_n^*\}_{n=1}^{\infty}$ have the property (*). Because of the maximal choice of c we infer that

$$\lim_{n\to\infty} d(G_n) = \lim_{n\to\infty} d(G_n^*) = c. \quad (***)$$

Consider now an arbitrary sequence of partitions $V_n = V_n^1 \cup V_n^2$, $n = 1, 2, \ldots$ having the property $||V_n^1| - |V_n^2|| \le 1$. Let G_n^1, G_n^2 be subgraphs of G_n, $n = 1, 2, \ldots$ induced on V_n^1 and V_n^2, respectively. We have

$$[d(G_n^1) + d(G_n^2)]\binom{\frac{v_n}{2}}{2} \ge c_n\binom{v_n}{2} - (1/2+\varepsilon_n)c_n\binom{v_n}{2}$$

and thus

$$d(G_n^1) + d(G_n^2) \ge (2 - 4\varepsilon_n)c_n\frac{v_n - 1}{v_n - 2} = c + o(1).$$

This, combined with (***) yields that the assumptions of the Lemma are satisfied and hence G_n contains K_k for every $n \ge n_k$. □

4. The Proof of Theorem 5

We will often use the following consequence of the Chernoff inequality [3] cf. 2.7 in [6]:

If $0 < p < 1$ and $0 < \alpha p < 1$ then

$$\sum \binom{m}{j} p^j (1-p)^{m-j} \leq \exp(mp(\alpha - 1) + \alpha pm \log 1/\alpha)$$

where the sum is over j such that $j \geq \alpha pm$ ($j \leq \alpha pm$) provided $\alpha > 1$ ($\alpha < 1$), resp.

Definition. Let G be a graph with vertex set $\{0, 1, \ldots, n\}$ and let $X \subset \{0, 1, \ldots, n\}$ be a set of cardinality $x + 1$ and p a real number, $0 < p < 1$.

We say that X has the property (a) if for every partition $X = X_1 \cup X_2$

(a) the number of edges of G which are subsets of either X_1 or X_2 is at least $0.74\frac{(x+1)(x-2)}{4}p$ and that X has the property (b) if for every partition $X = X_1 \cup X_2$

(b) the number of edges of G with one endpoint in X_1 and second in X_2 is at most $1.285p\frac{(x+1)(x-2)}{4}$. □

Consider a random graph R with vertex set $\{0, 1, \ldots, n\}$, where edges are chosen independently, each with probability p. We shall divide the proof into eight steps:

I) Let $X \subset \{0, 1, \ldots, n\}$, $|X| = x + 1$ be a given subset of the vertex set of R. Denote by $q_i(X)$ the probability that X fails to have property (i) ($i = a, b$) as $\binom{|X_1|}{2} + \binom{|X_2|}{2} \geq \frac{1}{4}(x+1)(x-2) = y$, we have

$$q_a(x) \leq 2^{x+1} \sum_{j<0.74yp} \binom{y}{j} p^j (1-p)^{y-j}$$

$$< 2^{x+1} \exp\left[\left(-0.26 + 0.74 \ln \frac{1}{0.74}\right) yp\right]$$

$$< \begin{cases} < \exp[(\ln 2 - 9{,}295 \cdot 10^{-3} p(x-2))(x+1)] \\ q_b(x) \leq 2^{x+1} \sum_{j>1.285py} \binom{y}{j} p^j (1-p)^{y-j} \\ < 2^{x+1} \exp\left[\left(0.285 + 1.285 \ln \frac{1}{1.285}\right) yp\right] \end{cases}$$

$$< \exp[(\ln 2 - 9.306 \cdot 10^{-3} p(x-2))(x+1)].$$

II) Let $p(n)$ denote the probability that the neighborhood N_k (in R) of a fixed vertex $k \in \{0, 1, \ldots, n\}$ has cardinality smaller than $0.97pn$.

We have

$$p(n) \leq \sum_{i<0.97pn} \binom{n}{i} p^i(1-p)^{n-i} < \exp[-4.5467 \cdot 10^{-4} np].$$

III) Let P_n denote the probability that the neighborhood of every vertex $k \in \{0, 1, \ldots, n\}$ have properties (a) and (b).

As $q_1(x)$ *and* $q_2(x)$ are clearly decreasing for

$$x > x_p = \frac{10^3 \ln 2}{p \,.\, 9.295} + 2,$$

we have for

$$0.97pn > x_0 \tag{18}$$

$$P_n > 1 - (n+1)[p(n) + q_a(0.97pn) + q_b(0.97pn)]. \tag{19}$$

IV) Let $s(n)$ denotes the probability that for a fixed pair of distinct vertices k, $l \in \{0, 1, \ldots, n\}$ there are more than $3p^2(n-1)$ triangles in R having common edge $\{k, l\}$.

We have

$$s(n) \leq p \sum_{j > 3p^2(n-1)} \binom{n-1}{j} p^{2j}(1-p^2)^{n-j-1} < p\exp(-1.295p^2(n-1)).$$

Thus, particularly for the probability S_n that every edge of R is contained in at most $3p^2(n-1)$ triangles we get

$$S_n \geq 1 - p\binom{n+1}{2}\exp(-1.295p^2(n-1)). \tag{20}$$

V) Let $k_l(G)$ denotes the number of complete l-gons in graph G. Then we have

$$E(k_4(R)) = \binom{n+1}{4} p^6$$

and

$$D(k_4(R)) = E(k_4^2(R)) - E^2(k_4(R))$$
$$\leq \binom{n+1}{8}\binom{8}{4} p^{12} + \binom{n+1}{7}\binom{7}{4} 4 \cdot p^{12} + \binom{n+1}{6}\binom{6}{4} \cdot 6 \cdot p^{11}$$
$$+ \binom{n+1}{5}\binom{5}{4}\binom{4}{3} p^9 + \binom{n+1}{4} p^6 - \binom{n+1}{4}^2 p^{12}.$$

Thus, according to Chebycheff inequality we have

$$1 - R_n \leq \operatorname{Prob}\left\{\left| k_4(R) - \binom{n+1}{4}p^6 \right| \geq 0.01\binom{n+1}{4}p^6\right\} \leq 100^2 \frac{D(X)}{\binom{n+1}{4}^2 p^{12}}$$
$$\leq \frac{1{,}6 \cdot 10^5}{n} + \frac{7{,}2 \cdot 10^5}{pn^2} + \frac{9{,}6 \cdot 10^5}{p^3(n^3-n)} + \frac{1}{\binom{n+1}{4}p^6}. \tag{21}$$

Similarly one can show

$$1 - Q_n = \text{Prob}\left\{\left|k_3(R) - p^3\binom{n+1}{3}\right| > 0.01p^3\binom{n+1}{3}\right\} < \frac{3{,}6\cdot 10^5}{n} + \frac{1{,}8\cdot 10^5}{pn^2}. \tag{22}$$

VI) Let $G = (V, E)$ be now a graph such that the neighborhood N_k has properties (a) and (b) for every $k \in \{0, 1, \ldots, n\}$. Consider a coloring $\psi\colon E \to \{1, 2\}$. Denote by T_1 the number of triangles which are monochromatic and by T_2 the number of triangles which are colored by two colors.

Then it follows by (a) and (b) that

$$3T_1 + T_2 \geq 0.37\cdot 3\cdot T$$
$$\text{and}\quad 2T_2 \leq 0.6425\cdot 3T$$

where $T = T(G)$ is the total number of triangles in G. It follows immediately from above inequalities that

$$T_1 \geq 0.04875T.$$

Moreover, if every edge G is contained in at most $3p^2(n-1)$ triangles (for real p, $0 < p < 1$) and G contains at least $0.99\binom{n+1}{3}p^3$ triangles then we get that the minimal number of edges that can destroy all monochromatic triangles is at least

$$\frac{0{,}04875}{3p^2(n-1)}T \geq \frac{0{,}04875\cdot 0.99\binom{n+1}{3}p^3}{3p^2(n-1)} \geq 2{,}681\cdot 10^{-3}n^2p.$$

VII) Suppose now that $k_4(G) \leq 1{,}01\binom{n+1}{4}p^6$. Destroy all 4-gons by removing at most $1.01\binom{n+1}{4}p^6$ edges.

$$\text{If } (1.01)\binom{n+1}{4}p^6 < 2{,}681\cdot 10^{-3}n^2p$$
$$\text{or equivalently } p^5 < 63{,}7\cdot 10^{-3}n^{-2} \tag{23}$$

then there are still some monochromatic triangles left and we are done.

VIII) Set

$$p = 0{,}576n^{-2/5}. \tag{24}$$

Then (23) holds. Note that (23) gets the following form

$$n^{1/5} > \frac{10^3\ln 2}{(0{,}576)^2 0{,}97\cdot 9{,}295} + 2 = 233{,}7 \tag{25}$$

which holds for $n + 1 = 7.02\cdot 10^{11}$

Now (19), (20), (21) and (22) imply, for the above choice of p and n that

$$P_n > 0{,}99$$
$$S_n > 0{,}99$$
$$R_n > 0{,}99$$
$$Q_n > 0{,}99$$

Hence, there exists a graph $G \in R$ (with $7{,}02 \cdot 10^{11}$ vertices) such that

α) neighborhood of any vertex has properties (a) and (b)

β) $k_4(G) < 1{,}01\binom{n+1}{4}p^6$

$k_3(G) > 0{,}99\binom{n+1}{3}p^3$

γ) every edge of G is contained in at most $3p^2(n-1)$ triangles

After deleting at most $1{,}01\binom{n+1}{4}p^6$ edges which destroy all complete 4-gons we get (as shown in VII) the desired graph. □

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References

1. Alon, N.: Personal communication
2. Bollobás, B.: Extermal Graph Theory. London-New York-San Francisco: Academic Press 1978
3. Chernoff, H.: A measure of asymptotic efficiency for tests of hypothesis based on the sum of observations. Ann. Stat. **23**, 493–502 (1952)
4. Erdös, P.: Problems and results on finite and infinite graphs. In: Recent Advances in Graph Theory (Proceedings of the Symposium held in Prague 1974) edited by M. Fiedler, pp 183–192. Praha: Academia 1974
5. Erdös, P.: On some of my conjectures in number theory and combinatorics. Congr. Numerantium **39**, 3–19 (1983)
6. Erdös, P., Spencer, J.: Probabilistic Methods in Combinatorics. Budapest: Akadémiai Kiado 1974
7. Folkman, J.: Graphs with monochromatic complete subgraphs in every edge colouring. SIAM J. Appl. Math. **18**, 19–29 (1970)
8. Goodman, A.W.: On sets of acquaintances and strangers at any party. Amer. Math. Mon. **66**, 778–783 (1959)
9. Graham, R.L.: On edgewise 2-coloured graphs with monochromatic triangles and containing no complete hexagon. J. Comb. Theory **4**, 300 (1968)
10. Graham, R.L.: Rudiments of Ramsey theory. Reg. Conf. Ser. Math. **45** (1981)
11. Irving, R.W.: On a bound of Graham and Spencer for a graph colouring constant. J. Comb. Theory (B) **15**, 200–203 (1973)
12. Nešetřil, J., Rödl, V.: The Ramsey property for graphs with forbidden complete subgraphs. J. Comb. Theory (B) **20**, 243–249 (1976)
13. Nešetřil, J., Rödl, V.: Partition theory and its applications. Lond. Math. Soc. Lect. Note Ser. **38**, 96–157 (1979)
14. Rödl, V.: On universality of graphs with uniformly distributed edges. Discrete Math. (to appear)

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