Topics in Euclidean Ramsey Theory

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1. Introduction

Many questions in Ramsey Theory can be placed in the following context. We are given a set X, a family $\mathcal{F}$ of distinguished subsets of X, and a positive integer r. We would like to decide whether or not the following statement holds: For any partition of $X = X_1 \cup \ldots \cup X_r$ into r classes, there is an $F \in \mathcal{F}$ and an index i such that $F \subseteq X_i$.

Such an F is usually called *homogeneous* (or *monochromatic*, if the partition of X is thought of as an r-coloring of X; we will use both terminologies interchangeably).

The key feature which distinguishes *Euclidean* Ramsey Theory from other branches of Ramsey Theory is the use of the Euclidean metric in determining the structure of $\mathcal{F}$. More precisely, X is usually taken to be Euclidean n-space $\mathbb{E}^n$ for some n, and $\mathcal{F} = \mathcal{F}(C)$ consists of all subsets F which are *congruent* to a given point set $C \subseteq \mathbb{E}^n$.

The requirement that the homogeneous set be congruent to C is quite stringent. For example, if C consists of three equally spaced collinear points then it turns out (as we shall see) that for any $n, \mathbb{E}^n$ can always be 4-colored with no monochromatic congruent copy of C formed, whereas monochromatic *homothetic* copies of C must always exist, as shown by van der Waerden's theorem, for example.

In this chapter I will survey some of the basic results in Euclidean Ramsey Theory as well as describing some very recent theorems and numerous open problems.

2. Preliminaries

Let us say that $R(C, n, r)$ holds if any r-coloring of $\mathbb{E}^n$ contains a monochromatic set congruent to C. Thus, for example, if C' is the set of three vertices of a unit equilateral triangle then $R(C', 4, 2)$ holds (by considering the five vertices of a unit simplex in $\mathbb{E}^4$) while $R(C', 2, 2)$ does not hold (by partitioning $\mathbb{E}^2$ into two classes of alternating strips of width $\sqrt{3/2}$, each open on the top and closed on the bottom.

Slightly more interesting is the following.

2.1 Theorem. *If S is the set of four vertices of a unit square then $R(S, 6, 2)$ holds.*

Proof. Consider the set $X \subseteq \mathbb{E}^6$ defined by $X = \{(x_1, \ldots, x_6) : x_i = 1/\sqrt{2}$ for exactly two values of i, and $x_i = 0$ for all other values of $i\}$. Any partition of $\mathbb{E}^6$ into two classes, say $\chi : \mathbb{E}^6 \to \{0, 1\}$, also partitions X into two classes. To each point $(x_1, \ldots, x_6) \in X$, we can associate a pair $\{i, j\}$ by letting i and j be the indices of the nonzero coordinates of $(x_1, \ldots, x_6)$. Thus, χ induces a 2-coloring of the edges of K_6, the complete graph on six vertices. It is a standard result in (Ramsey) graph theory that in any such 2-coloring, a monochromatic 4-cycle, say $c_1 \to c_2 \to c_3 \to c_4 \to c_1$, must be formed. It is now straightforward to check that this 4-cycle corresponds to the four vertices of a unit square in X, and the theorem is proved. □

It is no accident that in the examples we have given up to this point, proofs that $R(C, n, r)$ holds for some C were always accomplished by selecting only a suitable *finite* subset of $\mathbb{E}^n$ and coloring it (rather than all of $\mathbb{E}^n$). A standard compactness argument (see Graham, Rothschild, Spencer 1980) shows that this is always the case, although it is often far from obvious what the appropriate finite subset should be.

Before proceeding to more general considerations, we first discuss a tantalizing question which besides being among the most fundamental in the theory, illustrates quite clearly how little we [1] still know about what is going on in this area. For this example we take C to be the set C^* consisting of two points separated by distance 1.

To begin with, it is easy to see that $R(C^*, 2, 2)$ holds, simply by considering (as the suitable finite set) the set of three vertices of a unit equilateral triangle. To show that $R(C^*, 2, 3)$ holds, we need only consider the graph G (known as the Moser graph) shown in Fig. 1. Each edge $\{x, y\}$ of G denotes the fact that the distance between x and y is 1.

A simple calculation shows that the chromatic number of G is 4. Thus, any 3-coloring of $\mathbb{E}^2$ induces a 3-coloring of (infinitely many copies of) G and consequently, always produces a monochromatic pair of points at unit distance from each other, as claimed.

In the other direction, it is not difficult to 7-color the standard tiling of $\mathbb{E}^2$ by regular hexagons of side 9/10 so that no color class contains two points separated by distance 1. Thus, $R(C^*, 7, 2)$ does not hold. The least value d for which $R(C^*, d, 2)$ holds is also known as the *chromatic number* $\chi(\mathbb{E}^2)$ of $\mathbb{E}^2$, since it is the chromatic number of the (uncountable) graph formed by taking each point of $\mathbb{E}^2$ as a vertex and each pair $\{x, y\}$ with distance 1 between x

1 "we" meaning combinatorialists collectively, in this case.

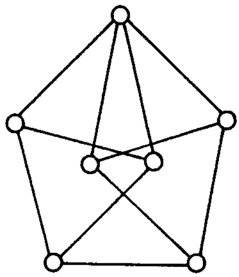

Fig. 1. The Moser graph

and y, as an edge. Thus, the best available bounds for $\chi(\mathbb{E}^2)$ are:

$$4 \leq \chi(\mathbb{E}^2) \leq 7.$$

There is some evidence that $\chi(\mathbb{E}^2) \geq 5$ from the result of Wormald (1979), who showed that $\mathbb{E}^2$ contains a (finite) graph of chromatic number 4, with all edges of length one and containing no 3–cycle and no 4–cycle.

For the chromatic number $\chi(\mathbb{E}^n)$ of $\mathbb{E}^n$, it has been recently shown by Frankl and Wilson (1981), using a powerful result on set systems with restricted intersections, that $\chi(\mathbb{E}^n)$ grows exponentially with n, verifying an earlier conjecture of Erdös. The best current bounds on $\chi(\mathbb{E}^n)$ are now:

$$(1 + o(1))(6/5)^n \leq \chi(\mathbb{E}^n) \leq (3 + o(1))^n.$$

3. Ramsey Sets

A basic concept in Euclidean Ramsey Theory is that of a Ramsey set.

Definition. A configuration C is said to be *Ramsey* if for all r there exists an $N = N(C, r)$ such that $R(C, N, r)$ holds.

An easy argument shows that no infinite set can be Ramsey. The following result forms the basis for constructing essentially all known Ramsey sets.

3.1 Theorem *(Erdös, Graham, Montgomery, Rothschild, Spencer, Straus 1973). If C_1 and C_2 are Ramsey then the cartesian product $C_1 \times C_2$ is Ramsey.*

Proof. Fix $C_1 \subseteq \mathbb{E}^m$, $C_2 \subseteq \mathbb{E}^n$ and let r be a positive integer. Choose u so that $R(C_1, u, r)$ holds. By the compactness theorem mentioned earlier there exists a finite set $T \subseteq \mathbb{E}^u$ such that in any r-coloring of T, a monochromatic congruent copy of C_1 is formed. Let $t = |T|$ and let $T = \{x_1, x_2, \ldots, x_t\}$. Choose v so that $R(C_2, v, r^t)$ holds (which is possible since C_2 is Ramsey). We claim that $R(C_1 \times C_2, u + v, r)$ holds. To prove the claim, suppose $\chi : \mathbb{E}^{u+v} \to \{1, 2, \ldots, r\}$ is an r-coloring of $\mathbb{E}^{u+v}$. Define an induced coloring $\chi' : \mathbb{E}^v \to \{1, 2, \ldots, r^t\}$ by

$$\chi'(y) = (\chi(x_1, y), \chi(x_2, y), \ldots, \chi(x_t, y)).$$

By the choice of v there is a χ'-monochromatic congruent copy of C_2, say $\overline{C}_2$, in $\mathbb{E}^v$. Now define an induced r-coloring χ'' of T by $\chi''(x_i) = \chi(x_i, y)$ for some $y \in \overline{C}_2$. This is well-defined since $\overline{C}_2$ is χ'-monochromatic. It is now straightforward to check that T contains a monochromatic (under the original coloring χ) copy of $C_1 \times C_2$. This proves the claim and consequently, the theorem follows. □

Since any two-point set is Ramsey then arbitrary cartesian products of two-point sets, i.e., the sets of vertices of rectangular parallelepipeds, are also Ramsey (and, of course, any *subset* of these sets of vertices). An interesting question which arises in this context is that of determining which simplexes (i.e., $(n+1)$-subsets of $\mathbb{E}^n$ in general position) are subsets of the vertex set of a rectangular parallelepiped. A *necessary* condition is that no angle determined by three of its vertices should exceed 90°. This condition turns out to be *sufficient* for $n = 2$ and $n = 3$. However, it is not sufficient for $n \geq 4$. Indeed, it is not difficult to construct a five-point simplex in $\mathbb{E}^4$ with all angles determined by three points being less than 89°, and which cannot be extended to the vertex set of any rectangular parallelepiped.

An ingenious construction recently discovered by Frankl and Rödl can be used to show that *any* set of three non-collinear points is Ramsey (thus partially resolving a conjecture in Graham (1980)). The idea behind their construction is the following. For arbitrary fixed k and r, and $n = n(k, r)$ chosen suitably large, consider the subset $X \subseteq \mathbb{E}^n$ formed as follows. For each subset $I \subseteq \{1, 2, \ldots, n\}$ of size $2k-1$, say $I = \{i_1, i_2, \ldots, i_{2k-1}\}$, define $x = x_I = (x_1, x_2, \ldots, x_n)$ by taking

$$x_j = \begin{cases} j & \text{if } j = i_u \text{ for } u = 1, 2, \ldots, k, \\ 2k - j & \text{if } j = i_u \text{ for } u = k+1, \ldots, 2k-1, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, a typical point x looks like:

$$x = (0, \ldots, 0, 1, 0, \ldots, 0, 2, 0, \ldots, 0, k, 0, \ldots, k-1, \ldots, 0, 1, 0, 0)$$

X is taken to be $\{x_I : I \subseteq \{1, \ldots, n\} \text{ with } |I| = 2k-1\}$. Consider now an arbitrary r-coloring of X. This induces an r-coloring of the set of all $(2k-1)$-subsets of $\{1, 2, \ldots, n\}$. Hence, if n is large enough then by Ramsey's Theorem there is a $(2k+1)$-subset $Y \subseteq \{1, 2, \ldots, n\}$ having all *its* $(2k-1)$-subsets in a single color. Suppose we write Y as $\{i_1, i_2, \ldots, i_{2k+1}\}$. Consider the three points x_{I_1}, x_{I_2} and x_{I_3} where

$$\begin{aligned} I_1 &= \{i_1, i_2, \ldots, i_{2k-1}\}, \\ I_2 &= \{i_2, i_3, \ldots, i_{2k}\}, \\ I_3 &= \{i_3, i_4, \ldots, i_{2k+1}\}. \end{aligned}$$

A straightforward calculation shows that distance(x_{I_1}, x_{I_2}) =distance(x_{I_2}, x_{I_3}) $= \sqrt{2k}$, distance$(x_{I_1}, x_{I_3}) = \sqrt{8k-2}$. Thus, x_{I_1}, x_{I_2} and x_{I_3} form an (arbitrarily) obtuse monochromatic isoceles triangle. Various obtuse triangles can now be formed from the sets of vertices of prisms created by taking the product of X with two-point sets. To form arbitrary obtuse triangles a similar technique is used, but with greater "shifts" of the $(1, 2, \ldots, k, \ldots, 2, 1)$ positions. Presumably, every non-degenerate simplex is Ramsey[2]. It would also be interesting to know whether such simple sets such as the five vertices of a regular pentagon are Ramsey but at present this is unknown[3].

As mentioned earlier the collinear set $C = \{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ with distance $(\mathbf{x}, \mathbf{y}) =$ distance$(\mathbf{y}, \mathbf{z}) = 1$ is *not* Ramsey. (Indeed, no set with three collinear points can be Ramsey, as we will see later). The proof of this is not difficult and goes as follows. For each point $\mathbf{u} \in \mathbb{E}^n$ assign the color

$$\chi(\mathbf{u}) = [\mathbf{u} \cdot \mathbf{u}] \pmod 4$$

where for $\mathbf{u} = (u_1, \ldots, u_n), \mathbf{u} \cdot \mathbf{u}$ denotes the inner product $\sum_{i=1}^{n} u_i^2$ and $[x]$ denotes the greatest integer not exceeding x. Suppose the set $C = \{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ occurs monochromatically in this 4-coloring of $\mathbb{E}^n$, say $C \subseteq \chi^{-1}(i)$. From Fig. 2, since

$$a^2 = b^2 + 1 - 2b \cos \Theta$$
$$c^2 = b^2 + 1 + 2b \cos \Theta$$

then

$$a^2 + c^2 = 2b^2 + 2$$

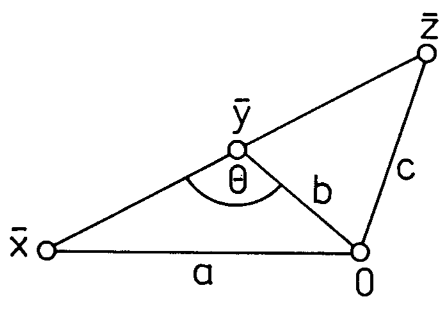

Fig. 2. The collinear set $C = \{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$

Since

$$\chi(\mathbf{x}) = \chi(\mathbf{y}) = \chi(\mathbf{z}) = i$$

then

2 This has now been proved by Frankl and Rödl (to appear).

3 The vertices of a regular pentagon do form a Ramsey set. This has been proved very recently by Igor Kříž (to appear).

$$
\begin{aligned}
a^2 &= 4k_a + i + e_a, \ 0 \le e_a < 1, \\
b^2 &= 4k_b + i + e_b, \ 0 \le e_b < 1, \\
c^2 &= 4k_c + i + e_c, \ 0 \le e_c < 1
\end{aligned}
$$

for suitable integers k_a, k_b, k_c. Thus, we have

$$4(k_a - 2k_b + k_c) - 2 = -e_a + 2e_b - e_c$$

which is easily seen to be (just barely) impossible. This proves that C is not Ramsey.

The preceding argument actually contains the kernel of an idea which when more fully developed leads to the following result.

Let us call a set $X \subseteq \mathbb{E}^n$ *spherical* if it is a subset of a sphere.

3.2 Theorem *(Erdős, Graham, Montogomery, Rothschild, Spencer, Straus 1973). If C is Ramsey then C is spherical.*

The proof, which we sketch for completeness, rests on several lemmas.

3.3 Lemma. *There exists a $(2n)$-coloring χ of $\mathbb{R}$ such that the equation*

$$\sum_{i=1}^{n}(y_i - y_i') = 1$$

has no solution with $\chi(y_i) = \chi(y_i')$, $1 \le i \le n$.

Proof. Define χ by setting $\chi(y) = j$ if $y \in [2m + j/n, 2m + (j+1)/n]$ for some integer m. Then $\chi(y_i) = \chi(y_i')$ implies

$$y_i - y_i' = 2m_i + \theta_i$$

for some θ_i with $|\theta_i| < 1/n$. Therefore

$$1 = \sum_{i=1}^{n}(y_i - y_i') = 2\sum_{i=1}^{n} m_i + \sum_{i=1}^{n} \theta_i = 2M + \theta$$

where $\theta = \sum_{i=1}^{n} \theta_i$. However, this is impossible since $0 \le |\theta| < 1$. □

3.4 Lemma *(Strauss 1975). Suppose $c_1, \ldots, c_n$ and $b \neq 0$ are arbitrary real numbers. Then there exists a $(2n)^n$-coloring χ^* of $\mathbb{R}$ such that the equation*

$$\sum_{i=1}^{n} c_i(x_i - x_i') = b \tag{1}$$

has no solution with $\chi^*(x_i) = \chi^*(x_i')$, $1 \le i \le n$.

Proof. Note that (1) holds if and only if

$$\sum_{i=1}^{n} c_i^*(x_i - x_i') = 1 \tag{2}$$

where $c_i^* = c_i b^{-1}$. Define χ^* on $\mathbb{R}$ by setting $\chi^*(\alpha) = \chi^*(\beta)$ if and only if $\chi(c_i^*\alpha) = \chi(c_i^*\beta)$ for all i, where χ is the $2n$-coloring defined in Lemma 3.3. Thus, χ^* is a $(2n)^n$-coloring. Now suppose (2) holds with $\chi^*(x_i) = \chi^*(x_i')$, $1 \leq i \leq n$. Then $\chi(c_j^* x_i) = \chi(c_j^* x_i')$, $1 \leq i, j \leq n$. In particular, $\chi(c_i^* x_i) = \chi(c_i^* x_i')$, $1 \leq i \leq n$. Therefore,

$$\begin{aligned}\sum_{i=1}^{n} c_i^*(x_i - x_i') &= \sum_{i=1}^{n}(c_i^* x_i - c_i^* x_i') \\ &= \sum_{i=1}^{n}(2m_i + \theta_i - 2m_i' - \theta_i') \\ &= 2M + \sum_{i=1}^{n}(\theta_i - \theta_i') \neq 1\end{aligned}$$

since $0 \leq \sum_{i=1}^{n} |\theta_i - \theta_i'| < 1$. □

3.5 Lemma. *A set $K = \{\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is not spherical if and only if there exist c_i, not all 0, such that:*
(i) $\sum_{i=1}^{k} c_i(\mathbf{v}_i - \mathbf{v}_0) = 0$,
(ii) $\sum_{i=1}^{k} c_i(\mathbf{v}_i \cdot \mathbf{v}_i - \mathbf{v}_0 \cdot \mathbf{v}_0) = b \neq 0$.

Proof. Assume K is a subset of a sphere with center $\mathbf{w}$ and radius r, and suppose K satisfies (i). By the law of cosines,

$$\begin{aligned}r^2 &= (\mathbf{v}_i - \mathbf{w}) \cdot (\mathbf{v}_i - \mathbf{w}) \\ &= (\mathbf{v}_0 - \mathbf{w}) \cdot (\mathbf{v}_0 - \mathbf{w}) + (\mathbf{v}_i - \mathbf{v}_0) \cdot (\mathbf{v}_i - \mathbf{v}_0) - 2(\mathbf{v}_i - \mathbf{v}_0) \cdot (\mathbf{w} - \mathbf{v}_0)\end{aligned}$$

which implies

$$(\mathbf{v}_i - \mathbf{v}_0) \cdot (\mathbf{v}_i - \mathbf{v}_0) = 2(\mathbf{v}_i - \mathbf{v}_0) \cdot (\mathbf{w} - \mathbf{v}_0)$$

since $(\mathbf{v}_0 - \mathbf{w}) \cdot (\mathbf{v}_0 - \mathbf{w}) = r^2$. Thus,

$$\sum_{i=1}^{k} c_i(\mathbf{v}_i - \mathbf{v}_0) \cdot (\mathbf{v}_i - \mathbf{v}_0) = 2(\mathbf{w} - \mathbf{v}_0) \cdot \sum_{i=1}^{k} c_i(\mathbf{v}_i - \mathbf{v}_0) = 0$$

which contradicts (ii).

On the other hand, suppose K is not spherical. We may assume without loss of generality that K is *minimally* non-spherical, i.e., all proper subsets of K are spherical. Thus, the $k+1$ points of K cannot form a simplex since a simplex is spherical. Therefore, the vectors $\mathbf{v}_i - \mathbf{v}_0$ are linearly dependent, i.e., there exist c_i, $1 \leq i \leq k$, not all 0, such that

$$\sum_{i=1}^{k} c_i(\mathbf{v}_i - \mathbf{v}_0) = 0. \tag{3}$$

By the minimality assumption on K, we may assume that $c_k \neq 0$ and that $\mathbf{v}_0, \ldots, \mathbf{v}_{k-1}$ lie on some sphere, say with center $\mathbf{w}$ and radius r. Since

$$\mathbf{v}_i \cdot \mathbf{v}_i - \mathbf{v}_0 \cdot \mathbf{v}_0 = (\mathbf{v}_i - \mathbf{w}) \cdot (\mathbf{v}_i - \mathbf{w}) - (\mathbf{v}_0 - \mathbf{w}) \cdot (\mathbf{v}_0 - \mathbf{w}) + 2(\mathbf{v}_i - \mathbf{v}_0) \cdot \mathbf{w}$$

then

$$\begin{aligned}\sum_{i=1}^{k} c_i(\mathbf{v}_i \cdot \mathbf{v}_i - \mathbf{v}_0 \cdot \mathbf{v}_0) &= \sum_{i=1}^{k} c_i((\mathbf{v}_i - \mathbf{w}) \cdot (\mathbf{v}_i - \mathbf{w}) - (\mathbf{v}_0 - \mathbf{w}) \cdot (\mathbf{v}_0 - \mathbf{w})) \\ &\quad + 2\sum_{i=1}^{k} c_i(\mathbf{v}_i - \mathbf{v}_0) \cdot \mathbf{w} \\ &= c_k((\mathbf{v}_k - \mathbf{w}) \cdot (\mathbf{v}_k - \mathbf{w}) - r^2) \neq 0\end{aligned}$$

by (3) since $\mathbf{v}_k$ is *not* on the sphere of radius r centered at $\mathbf{w}$. Thus (ii) holds and the lemma is proved. □

We are now ready to complete the proof of Theorem 3.2. Assume $C = \{\mathbf{v}_0, \ldots, \mathbf{v}_n\}$ is not spherical. By Lemma 3.5, there exist $c_1, c_2, \ldots, c_n$ and $b \neq 0$ such that

$$\sum_{i=1}^{n} c_i(\mathbf{v}_i - \mathbf{v}_0) = 0, \ \sum_{i=1}^{n} c_i(\mathbf{v}_i \cdot \mathbf{v}_i - \mathbf{v}_0 \cdot \mathbf{v}_0) = b \neq 0. \tag{4}$$

Let us color each point $\mathbf{u}$ of $\mathbb{E}^N$ with χ by defining $\chi(\mathbf{u}) = \chi^*(\mathbf{u} \cdot \mathbf{u})$ where χ^* is the $(2n)^n$-coloring used in Lemma 3.4 with these values of c_i and b. Thus, if χ assigns a single color to all the $\mathbf{v}_i$ then χ^* must assign a single color to all the $\mathbf{v}_i \cdot \mathbf{v}_i$. However, this is impossible since (4) cannot hold monochromatically using the coloring χ^*. Thus, with the $(2n)^n$-coloring χ of $\mathbb{E}^N$ given above, the set C cannot occur monochromatically. Since N was arbitrary, this shows that C is not Ramsey, and the theorem is proved. □

Before concluding this section we point out that a number of analogues to the preceding results are known when instead of requiring a monochromatic set congruent to the given set C, we only require that the congruent set have at most k colors for some fixed value of k. Specifically, call a configuration *k-Ramsey* if for any r there is an $N = N(k, C, r)$ such that in any r-coloring of $\mathbb{E}^N$, some set congruent to C must occur which has at most k colors. Thus, 1-Ramsey sets are just Ramsey sets. The following analogue to Theorem 3.2 appears in Erdős et al. (1973).

3.6 Theorem. *If C is k-Ramsey then C is contained in the union of k spheres.*

4. Sphere-Ramsey Sets

Rather than take all of $\mathbb{E}^n$ as our underlying space, it is possible to consider various subsets of $\mathbb{E}^n$ instead and ask the analogous questions. A very natural choice for such subsets are unit spheres. Specifically, we denote by S^n the unit sphere in $\mathbb{E}^{n+1}$ centered at the origin, i.e.,

$$S^n = \{(x_0,\ldots,x_n) \in \mathbb{E}^{n+1} : \sum_{i=0}^{n} x_i^2 = 1\}.$$

A configuration C will then be called *sphere-Ramsey* if for any r, there is an $N = N(C,r)$ such that in any r-coloring of S^N there is always a monochromatic subset of S^N which is congruent to C. In this section we will describe several results concerning sphere-Ramsey sets which bear some similarity to those for ordinary Ramsey sets, although in general far less is known about sphere-Ramsey sets.

The strongest constraint currently known for sphere-Ramsey sets is given by the following result.

4.1 Theorem *(Graham 1983). If $X = \{\mathbf{x}_1,\ldots,\mathbf{x}_m\} \subseteq \mathbb{E}^n$ is sphere-Ramsey then for any linear dependence $\sum_{i\in I} \alpha_i \mathbf{x}_i = \mathbf{0}$ there must exist a nonempty subset $J \subseteq I$ such that $\sum_{j\in J} \alpha_j = 0$.*

Proof. Suppose the contrary, i.e., suppose

(i) for some nonempty $I \subseteq \{1,2,\ldots,m\}$, there exist nonzero α_i, $i \in I$, such that
$$\sum_{i\in I} \alpha_i \mathbf{x}_i = \mathbf{0};$$

(ii) for all nonempty $J \subseteq I$,
$$\sum_{j\in J} \alpha_j \neq 0.$$

We will show that there exists an $r = r(X)$ such that for any N, S^N can be r-colored with no monochromatic subset congruent to X.

To begin with, consider the homogeneous linear equation

$$\sum_{i\in I} \alpha_i z_i = 0. \tag{5}$$

By assumption (ii), Rado's results for the partition regularity of this equation over $\mathbb{R}^+$ (see Graham, Rothschild, Spencer 1980 or Rado 1933) implies that (5) is *not* regular, i.e., for some r there is an r-coloring $\chi : \mathbb{R}^+ \to \{1,2,\ldots,r\}$ such that (5) has no monochromatic solution. Color the points of

$$S_+^N = \{(x_0,\ldots,x_N) \in S^N : x_0 > 0\}$$

with χ^* by defining

$$\chi^*(\mathbf{x}) = \chi(\mathbf{u} \cdot \mathbf{x})$$

where $\mathbf{u}$ denotes the unit vector $(1, 0, \ldots, 0)$. Thus, the color of $\mathbf{x} \in S_+^N$ depends only on the distance of $\mathbf{x}$ to the "north pole" of S^N.

For each nonempty subset $J \subseteq I$, consider the equation

$$\sum_{j \in J} \alpha_j z_j = 0. \tag{6}$$

Of course, by (ii) this equation also fails to satisfy the necessary and sufficient condition of Rado for partition regularity. Therefore, there is an r_J-coloring χ_J of $\mathbb{R}^+$ so that (6) has no χ_J-monochromatic solution. As before, we can color S_+^N by giving $\mathbf{x} \in S_+^N$ the color

$$\chi_J^*(\mathbf{x}) = \chi_J(\mathbf{u} \cdot \mathbf{x}).$$

Now, we form the product coloring $\hat{\chi}$ of S_+^N by defining for $\mathbf{x} \in S_+^N$

$$\hat{\chi}(\mathbf{x}) = (\ldots, \chi_J(\mathbf{x}), \ldots)$$

where the index J ranges over all $2^{|I|} - 1$ nonempty subsets of I. The number of colors required by the coloring $\hat{\chi}$ is at most

$$R = \prod_{\phi \neq J \subseteq I} r_J.$$

An important property of $\hat{\chi}$ is this. Suppose we extend $\hat{\chi}$ to

$$S_0^N = \{(x_0, \ldots, x_n) \in S^N : x_0 \geq 0\}$$

by assigning *all* R colors to any point in $S_0^N \backslash S_+^N$, i.e., with $x_0 = 0$. Then the only monochromatic solution to (5) in $\mathbb{R}^+ \cup \{0\}$ is $z_i = 0$ for all $i \in I$.

Next, we construct a similar coloring χ on $S_-^N = \{-\mathbf{x} : \mathbf{x} \in S_+^N\}$, but using R different colors. This assures that any set X which intersects both hemispheres S_+^N and S_-^N cannot be monochromatic.

Finally, we have left to color the equator

$$S^{N-1} = \{\mathbf{x} \in S^N : x_0 = 0\}.$$

By the construction, any monochromatic set congruent to X must be contained entirely in S^{N-1}. Hence, it suffices to color S^{N-1} avoiding monochromatic copies of X, where we may use any of $2R$ colors previously used in the coloring of $S_+^N \cup S_-^N$. By induction, this can be done provided we can so color S^1. However, if $m > 1$ then S^1 can in fact always be 3-colored without a monochromatic copy of X. This completes the proof of the theorem. □

4.2 Corollary. *If $X \subseteq S^n$ and $\mathbf{0} \in$ conv(X) then X is not sphere-Ramsey (where conv(X) denotes the convex hull of X).*

consists of the $\binom{2N}{N}$ N–element subsets of $\{1,\ldots,2N\}$. If $F,F' \in \mathcal{F}$, $F \neq F'$, then

$$|F \cap F'| \equiv N \ (mod\, q)$$

if and only if

$$|F \cap F'| = N - q = eq.$$

If the elements of $\mathcal{F}$ are r-colored then some color class must contain at least

$$\frac{1}{r}|\mathcal{F}| = \frac{1}{r}\binom{2N}{N} > \binom{2N}{q-1}$$

elements of $\mathcal{F}$. However, by the preceding result of Frankl and Wilson, if $|F \cap F'| = eq$ never occurs, then the number of elements of $\mathcal{F}$ can be at most $\binom{2N}{q-1}$, which is a contradiction.

Therefore, $\mathcal{F}$ must contain a monochromatic pair $F(\mathsf{s}), F(\mathsf{s}')$ with

$$|F(\mathsf{s}) \cap F(\mathsf{s}')| = eq.$$

This implies that s and s' must (up to a permutation of coordinates) look like:

$$\begin{array}{rlllll} & & eq & q & q & eq \\ \mathsf{s} = (\alpha, & & \beta,\ldots,\beta, & \beta,\ldots,\beta, & -\beta,\ldots,-\beta, & -\beta,\ldots,-\beta), \\ \mathsf{s}' = (\alpha, & & \beta,\ldots,\beta, & -\beta,\ldots,-\beta, & \beta,\ldots,\beta, & -\beta,\ldots,-\beta). \end{array}$$

It now follows that

$$\text{distance}(\mathsf{s},\mathsf{s}') = \sqrt{8q\beta^2} = 2\lambda$$

and

$$\text{distance}(\mathsf{s},\mathbf{0}) = \text{distance}(\mathsf{s}',\mathbf{0}) = \alpha^2 + 2(1+e)q\beta^2 = 1.$$

Thus, s and $\mathsf{s}' \in S^{2N}$ and the theorem is proved. □

5. Concluding Remarks

Space limitations have prevented us from describing more than just a few of the many interesting results and problems in Euclidean Ramsey Theory. Several topics we might have discussed are the following.

Let us call a collection C of *line segments* in $\mathbb{E}^n$ *line-Ramsey* if for any r, in any partition of all the line segments in $\mathbb{E}^n$ into r classes, some class contains a set of line segments congruent to C. It is known (Erdös et al. 1973), for example, that if C is line-Ramsey then all line segments must have the same length. Another negative result is the following.

5.1 Theorem *(Graham 1983). Suppose C is a configuration of unit line segments L_i such that:*

(i) The set of endpoints of the L_i is not spherical;

(ii) The graph having the L_i as its edges is not bipartite.

Then C is not line-Ramsey.

Proof. If $\mathbf{0} \in conv(X)$ then there exist $\alpha_x > 0$, $\mathbf{x} \in X' \subseteq X$, such that

$$\sum_{\mathbf{x} \in X'} \alpha_{\mathbf{x}} \mathbf{x} = \mathbf{0}.$$

Since no subset of the $\alpha_{\mathbf{x}}$ can sum to 0, the result follows. □

In the other direction, it is known that the vertex set of any rectangular parallelepiped is sphere-Ramsey, provided the length of its main diagonal is at most $\sqrt{2}$. The proof has the same basic structure as the usual proofs of the Hales-Jewett theorem and can be found in Graham (1983). It seems likely that this should hold in fact for any rectangular parallelepiped with main diagonal length less than 2[4]. Here, we show this for the case of two points. Specifically, we have

4.3 Theorem. *For any λ with $0 < \lambda < 1$, the set $\{-\lambda, \lambda\}$ is sphere-Ramsey.*

Proof. It is enough to show that the graph $G(\lambda)$ with vertex set S^n and edge set $\{\{\mathbf{x}, \mathbf{y}\} : \text{ distance } (\mathbf{x}, \mathbf{y}) = \lambda\}$ has chromatic number tending to infinity with n. To prove this we use the following result of Frankl and Wilson:

4.4 Theorem *(Frankl, Wilson 1981). Let $\mathcal{F}$ be a family of k-sets of $\{1, 2, \ldots, n\}$ such that for some prime power q,*

$$|F \cap F'| \not\equiv k (mod\, q)$$

for all $F \neq F'$ in $\mathcal{F}$. Then

$$|\mathcal{F}| \leq \binom{n}{q-1}.$$

For a fixed r, choose a prime power q so that

$$\binom{2(1+e)q}{(1+e)q} > r \binom{2(1+e)q}{q-1}$$

where $\beta = \lambda / \sqrt{2q}$, and α and $e > 0$ are chosen so that

$$\alpha^2 + 2(1+e)q\beta^2 = 1$$

and $N = (1+e)q$ is an integer. Consider the set

$$S = \{(s_0, \ldots, s_{2N}) : s_0 = \alpha,\ s_i = \pm\beta, \sum_{i=1}^{2N} s_i = 0\}.$$

To each $\mathbf{s} \in S$ associate the subset

$$F(\mathbf{s}) = \{i \in \{1, \ldots, 2N\} : s_i = \beta\}.$$

Thus, the family

$$\mathcal{F} = \{F(\mathbf{s}) : \mathbf{s} \in S\}$$

4 This has now been proved by Frankl and Rödl (to appear).

It is not known whether four line segments forming a unit square is line-Ramsey.

Even if we restrict ourselves to $\mathbb{E}^2$, there are many unsolved problems. For example, is it true that if T is any three-point set in $\mathbb{E}^2$ which does not form an equilateral triangle, then $R(T,2,2)$ holds? The strongest conjecture would be that in *any* 2-coloring of $\mathbb{E}^2$, a congruent copy of *every* three point set must occur monochromatically, with the exception of the set of vertices of a single equilateral triangle. On the other hand, it may be true that $R(T,2,3)$ *never* holds for *any* three-point set T.

Since we have seen that $\mathbb{E}^2$ can be 7-colored so that no set congruent to a given two-point occurs monochromatically, one might wonder if there were any interesting Euclidean Ramsey properties which hold when $\mathbb{E}^2$ is partitioned into an arbitrarily large (finite) number of colors. The following result shows that there are.

5.2 Theorem *(Graham 1980). For every partition of $\mathbb{E}^2$ into finitely many classes, some class has the property that for all $\alpha > 0$, it contains three points which span a triangle of area α.*

The proof, which can be extended to the analogous result for $\mathbb{E}^n$, is surprisingly tricky.

For other Euclidean Ramsey Theory results, the reader can consult Erdös et al. (1973), Erdös, Rothschild, Straus (1983), Graham (1980), Graham (1983), Shader (1974, 1976), Straus (1975).

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