Van Der Warden's (VDW) Thm

Exposition by William Gasarch

July 19, 2024

VDW's Thm

Def Let $W, k, c \in \mathbb{N}$. Let $\mathrm{COL}\colon [W] \rightarrow [c]$. A **mono k-AP** is an arithmetic progression of length k where every elements has the same color. We often say

$$a, a+d, \ldots, a+(k-1)d \text{ are all he same color}$$

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We will determine W later.
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1. Within one block.
2. If we take enough blocks, how they relate.

Within a Block

Def: $a, a+d, a+2d$ is an **almost mono 3AP** if $\mathrm{COL}(a) = \mathrm{COL}(a+d) \ne \mathrm{COL}(a+2d)$. The **color of an almost mono 3AP** is $\mathrm{COL}(a) = \mathrm{COL}(a+d)$.

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Look at the first three elements of a block of 5:

1. RRR or BBB. 1-2-3 is mono 3AP.
2. RBR or BRB. 1-3-5 is mono 3AP or almost mono 3AP.
3. RBB or BRR. 2-3-4 is mono 3AP or almost mono 3AP.
4. BBR or RRB. 1-2-3 is almost mono 3AP.
5. BRB. 1-3-5 is a mono 3AP or an almost mono 3AP.
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So always get a mono 3AP or an almost mono 3AP. Can assume its almost mono 3AP and its R.

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Hence need to take $W = 5 \times 65 = 365$.

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We can get by with LESS blocks- we will consider this point after the proof.

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- In each block there is a mono 3AP or an almost mono 3AP. (This is why blocks-of-5.)

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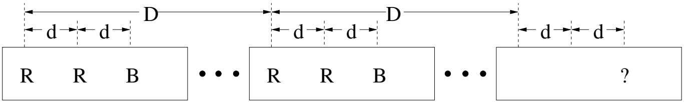

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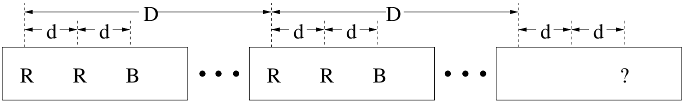

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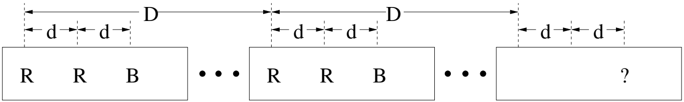

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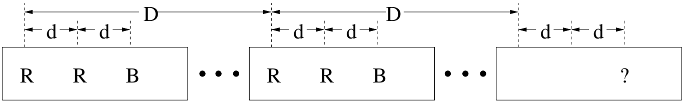

If ? is B then get B 3-AP.
If ? is R then get R 3-AP.
Done!

Side Note: Can Get By With Less Blocks

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So we don't really have to look at 32 colorings.

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How many colorings of a block already have a mono 3AP.

Side Note: Can Get By With Less Blocks (cont)

RRRXY with $X, Y \in \{R, B\}$. 4 colorings.
BBBXY with $X, Y \in \{R, B\}$. 4 colorings.
RBRRR
RBRBR
BRBBB
BRBRB
RBBBX with $X \in \{R, B\}$. 2 colorings.
BRRRX with $X \in \{R, B\}$. 2 colorings.
RRBBB
BBRRR

Side Note: Can Get By With Less Blocks (cont)

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There are 16 blocks which already have a mono 3AP. Hence can use $32 - 16 = 16$ blocks.

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I really do not care.

Is $W(3,2) = 365$?

No
What is $W(3,2)$?

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One can work out by hand that

$$W(3,2) = 9.$$

We will later say which VDW numbers are know and how they compare to the bounds given by the proof of VDW's Thm.

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Spoiler Alert The few known VDW numbers are **much smaller** than the bounds given by the proof of VDW's Thm.

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How big should the blocks be?

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Need for all 3^7 colorings of blocks get a mono 3AP or an almost mono 3AP.
Need $2 \times 3^{+}1$ blocks.

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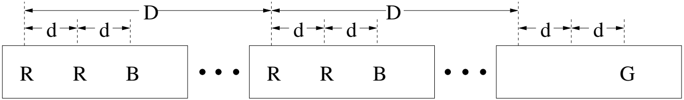

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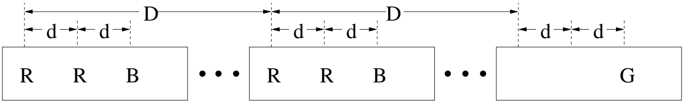

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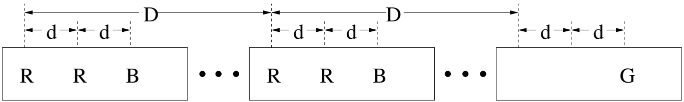

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We have 2 almost mono 3APs of diff colors that same last element.

I Like Big Blocks and I Cannot Lie!

Let W be LOTS of blocks of size $7 \times 2 \times (3^7 + 1)$.

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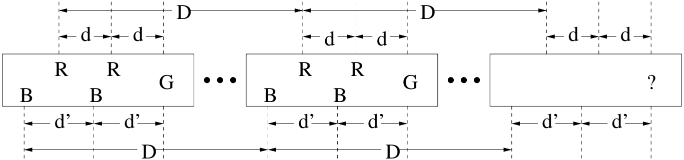

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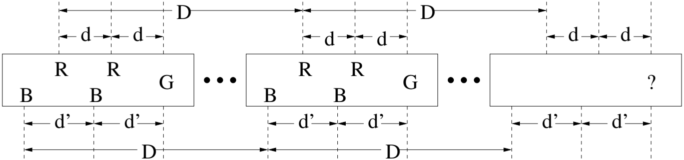

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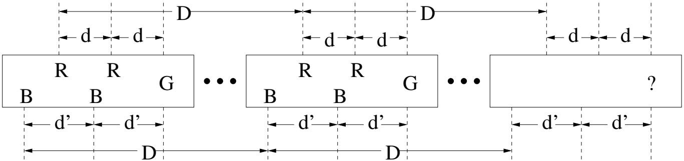

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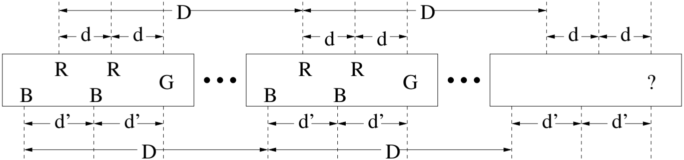

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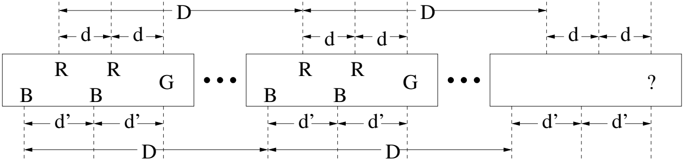

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$W(3, c)$

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- You COULD do a proof that $W(3,4)$ exists. You would need to iterate what I did twice.
- You can BELIEVE that $W(3,c)$ exists though might wonder how to prove it formally.

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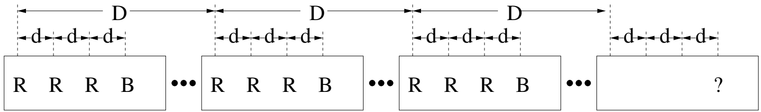

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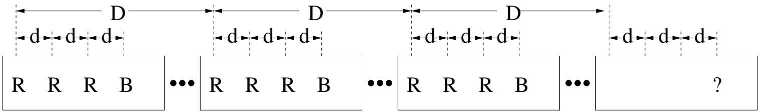

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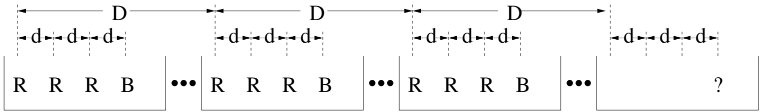

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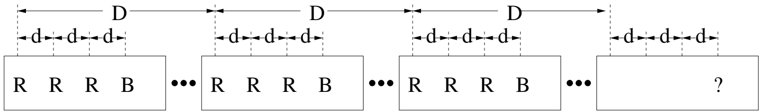

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