BILL, RECORD LECTURE!!!!

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Van Der Warden's (VDW) Thm

Exposition by William Gasarch

January 23, 2025

VDW's Thm

Def Let $W, k, c \in \mathbb{N}$. Let $\mathrm{COL}\colon [W] \to [c]$. A **mono** ***k*-AP** is an arithmetic progression of length k where every elements has the same color. We often say

$$a, a+d, \ldots, a+(k-1)d \text{ are all he same color}$$

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VDW's Thm For all k, c there exists $W = W(k, c)$ such that for all $\mathrm{COL}\colon [W] \to [c]$ there exists a mono k-AP.

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We will determine W later.
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1. Within one block.
2. If we take enough blocks, how they relate.

Within a Block

Def: $a, a+d, a+2d$ is an **almost mono 3AP** if $\mathrm{COL}(a) = \mathrm{COL}(a+d) \neq \mathrm{COL}(a+2d)$. The **color of an almost mono 3AP** is $\mathrm{COL}(a) = \mathrm{COL}(a+d)$.

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Look at the first three elements of a block of 5:

1. RRR or BBB. 1-2-3 is mono 3AP.
2. RBR or BRB. 1-3-5 is mono 3AP or almost mono 3AP.
3. RBB or BRR. 2-3-4 is mono 3AP or almost mono 3AP.
4. BBR or RRB. 1-2-3 is almost mono 3AP.
5. BRB. 1-3-5 is a mono 3AP or an almost mono 3AP.
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So always get a mono 3AP or an almost mono 3AP. Can assume its almost mono 3AP and its R.

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Hence need to take $W = 5 \times 65 = 365$.

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Hence need to take $W = 5 \times 65 = 365$.
We can get by with LESS blocks- we will consider this point after the proof.

$W(3,2) \le 365$

Let $COL\colon [W] \to [2]$.

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Break $[W]$ into 65 blocks of size 5 which we think of as being 32-colored.

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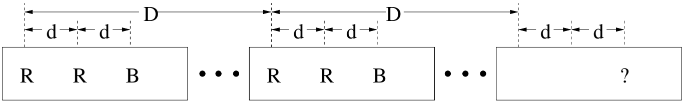

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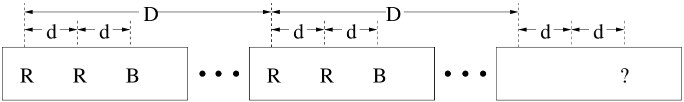

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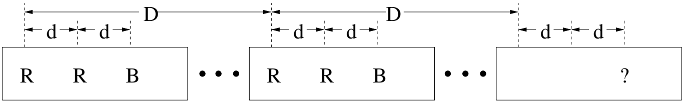

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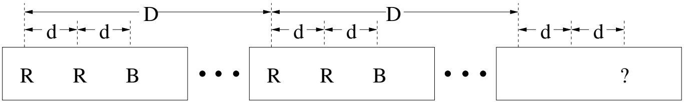

If ? is B then get B 3-AP.
If ? is R then get R 3-AP.
Done!

Side Note: Can Get By With Less Blocks

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So we don't really have to look at 32 colorings.

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How many colorings of a block already have a mono 3AP.

Side Note: Can Get By With Less Blocks (cont)

RRRXY with $X, Y \in \{R, B\}$. 4 colorings.
BBBXY with $X, Y \in \{R, B\}$. 4 colorings.
RBRRR
RBRBR
BRBBB
BRBRB
RBBBX with $X \in \{R, B\}$. 2 colorings.
BRRRX with $X \in \{R, B\}$. 2 colorings.
RRBBB
BBRRR

Side Note: Can Get By With Less Blocks (cont)

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There are 16 blocks which already have a mono 3AP. Hence can use $32 - 16 = 16$ blocks.

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I really do not care.

Is $W(3,2) = 365$?

No

What is $W(3,2)$?

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One can work out by hand that

$$W(3,2) = 9.$$

We will later say which VDW numbers are know and how they compare to the bounds given by the proof of VDW's Thm.

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Spoiler Alert The few known VDW numbers are **much smaller** than the bounds given by the proof of VDW's Thm.

$W(3,3)$

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How big should the blocks be?

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Need for all 3^7 colorings of blocks get a mono 3AP or an almost mono 3AP.
Need $2 \times 3^{+}1$ blocks.

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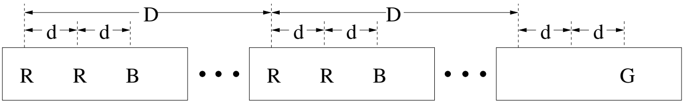

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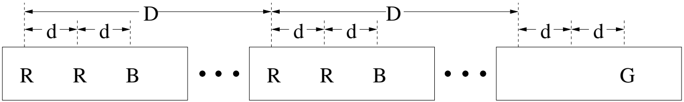

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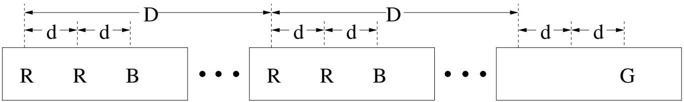

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We have 2 almost mono 3APs of diff colors that same last element.

I Like Big Blocks and I Cannot Lie!

Let W be LOTS of blocks of size $7 \times 2 \times (3^7 + 1)$.

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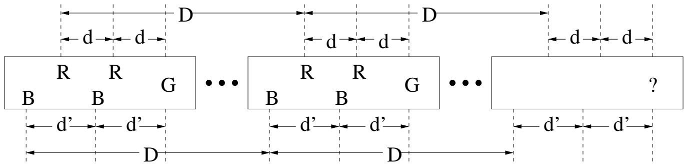

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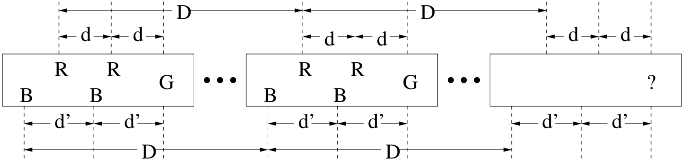

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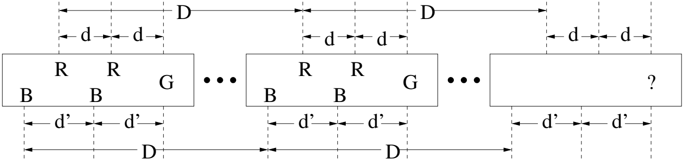

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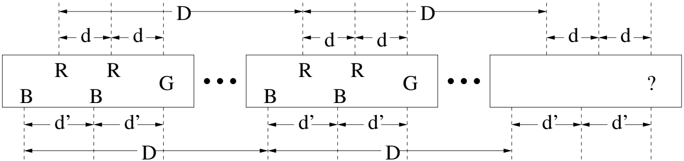

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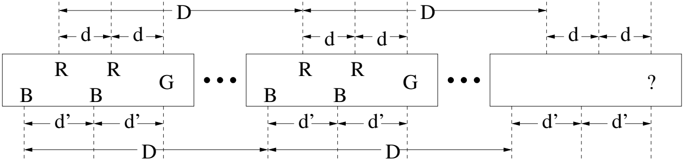

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From what you have seen:

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Note that we **do not** do
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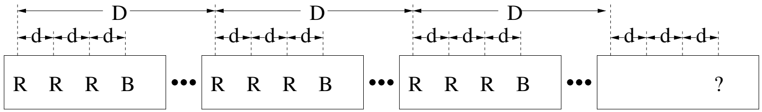

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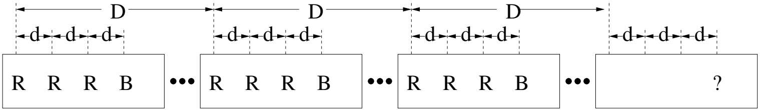

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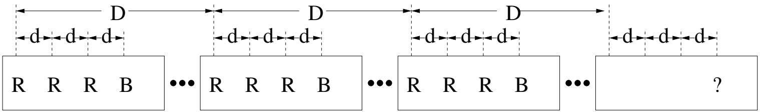

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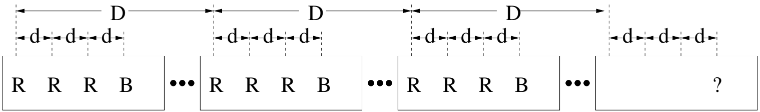

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