Are There Better Bounds on the VDW Numbers?

Exposition by William Gasarch

July 16, 2024

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Logician (Shelah) proved W(k, c) prim rec: clever!

- Proof is elementary. Can present here but won't.
- Bounds still large. Fifth Level of PR hierarchy.

Deep Math From Search for Better Upper Bounds on VDW Numbers

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It DID succeed! (Oh! Thats a good thing!)

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2. $\{x^2 : x \in \mathbb{N}\}$ has upper den 0.

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The hope was that the proof of Conj would require a new proof of VDW's Theorem that would lead to better bounds.

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- Roth won the Fields Medal in 1958 for his work on Diophantine approximation (so not for this work).

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  - Causes of change: (1) combinatorics using deep math, (2) CS inspired new problems in combinatorics.

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None of these results used mathematics of interest.

Known Lower Bounds

1. Easy Use of Prob Method $W(k,2) \ge \sqrt{k}2^{k/2}$ (Easy extension to 3 colors)
2. Very sophisticated use yields $W(k,2) \ge \frac{2^k}{k^\epsilon}$ (Does not extend to 3 colors.)
3. If p is prime then $W(p,2) \ge p(2^p - 1)$. Constructive! (Does not extend to 3 colors.)

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- There is also a ConservaMedical Medal- an alternative to the Nobel Prize in Medicine. It went to Donald Trump for his Medical Advice on Covonavirus. I am kidding.