BILL, RECORD LECTURE!!!!

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Small Ramsey Numbers

Exposition by William Gasarch

March 12, 2025

The First Theorem in Ramsey Theory

Thm For all $\mathrm{COL}\colon \binom{[6]}{2} \to [2]$ there exists a homog set of size 3.

Focus on Vertex 1

Given a 2-coloring of the edges of K_6 we look at vertex 1.

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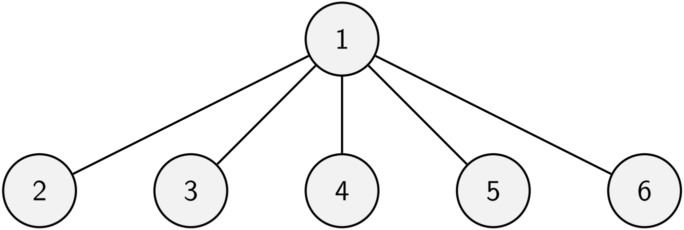

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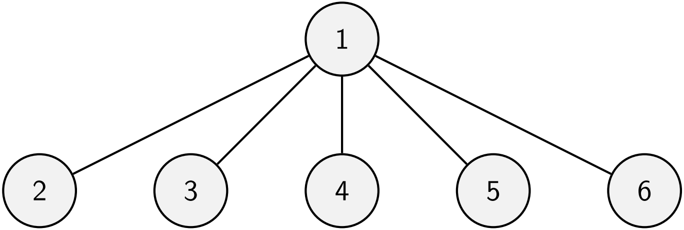

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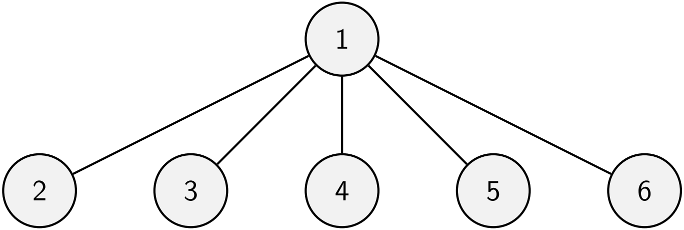

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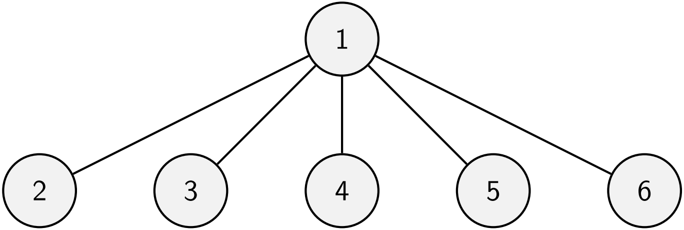

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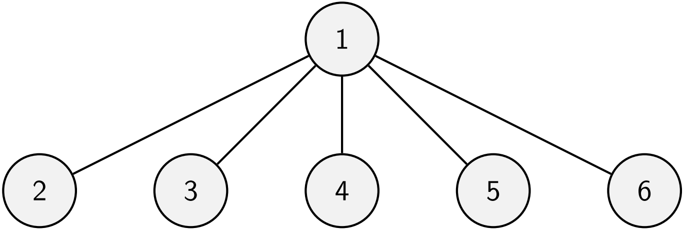
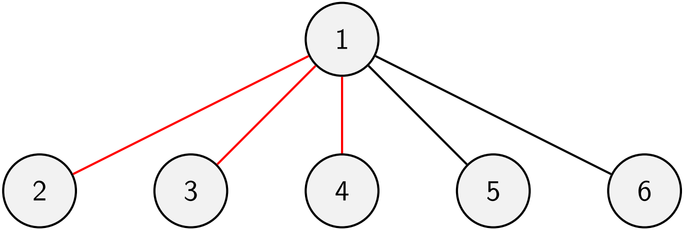
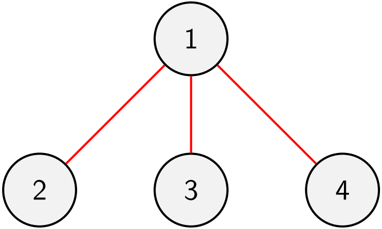


There are 5 edges coming out of vertex 1.
They are 2 colored.
$\exists$ 3 edges from vertex 1 that are the same color.
We can assume $(1,2)$, $(1,3)$, $(1,4)$ are all **RED**.

(1,2), (1,3), (1,4) are RED



We Look Just at Vertices 1,2,3,4



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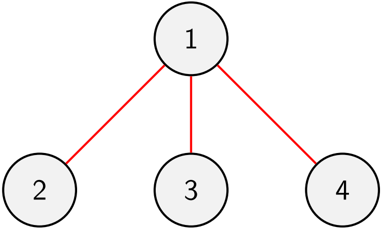

If $(2,3)$ is **RED** then get **RED** Triangle. So assume (2,3) is **BLUE**.

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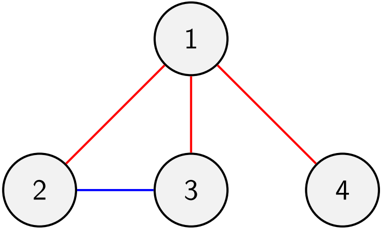

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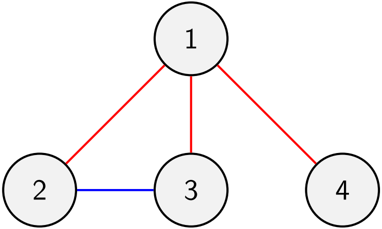

If (3,4) is **RED** then get **RED** triangle. So assume (3,4) is **BLUE**.

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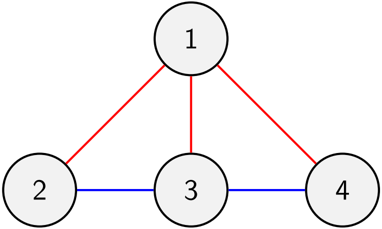

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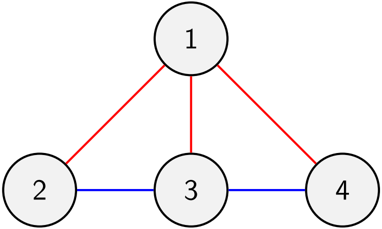

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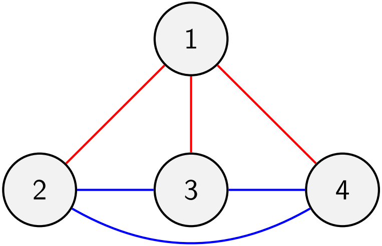

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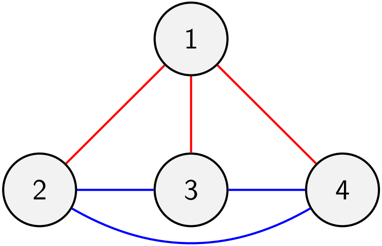

Note that there is a **BLUE** triangle with verts $2, 3, 4$. Done!

What If We Color Edges Of K_5?

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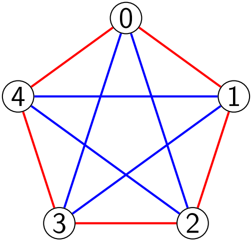

This graph is not arbitrary.
$\mathrm{SQ}_5 = \{x^2 \pmod 5 : 0 \le x \le 4\} = \{0, 1, 4\}$.

- If $i - j \in \mathrm{SQ}_5$ then **RED**.
- If $i - j \notin \mathrm{SQ}_5$ then **BLUE**.

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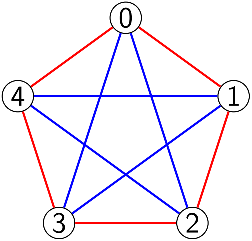

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Two ways to show no mono $\triangle$s on next slide.

No Mono $\triangle$s

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Need to show there are no mono $\triangle$.

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The **R** edges are all symmeric so if there is a $\triangle$ edge can assume one of the edges is $(0, 1)$. No x with $\mathrm{COL}(0, x) = \mathrm{COL}(0, 1) =$ **R**.

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$x^2 + y^2 + z^2 \equiv 0$.

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UPSHOT $R(3,3) = 6$.

Asymmetric Ramsey Numbers

Definition Let $a, b \geq 2$. $R(a, b)$ is least n such that for all 2-colorings of K_n there is **either** a red K_a or a blue K_b.

1. $R(a, b) = R(b, a)$.
2. $R(2, b) = b$
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Proof left to the reader, but its easy.

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The proof has three cases on the next three slides.

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1. There is a vertex with large **Red** Deg.

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1. There is a vertex with large **Red** Deg.
2. There is a vertex with large **Blue** Deg.
3. All verts have small **Red** degree and small **Blue** degree.

Some Vertex v Has Large Red Deg

Case 1 $(\exists v)[\deg_R(v) \geq R(a-1,b)]$.

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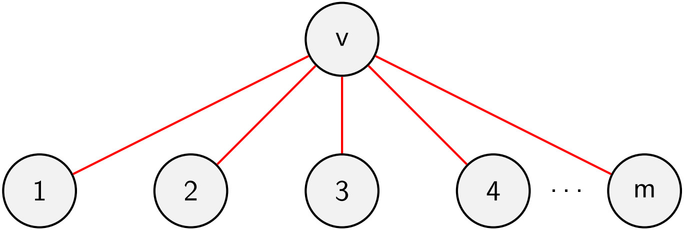

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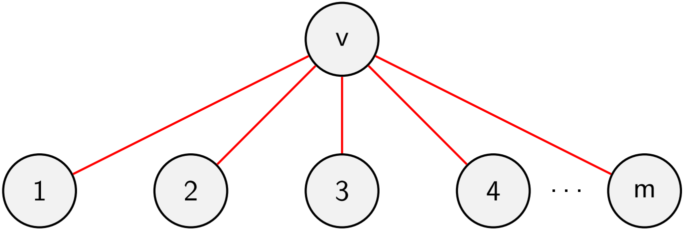

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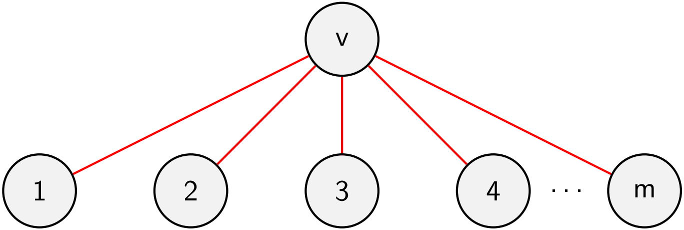

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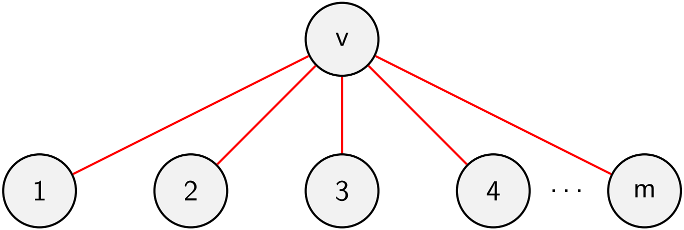

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Case 1.2 There is a **Blue** K_b in $\{1,\ldots,m\}$. DONE.
Case 1.3 Neither. **Impossible** since $m = R(a-1,b)$.

Some Vertex v Has Large Blue Deg

Case 2 $(\exists v)[\deg_B(v) \ge R(a, b-1)]$.

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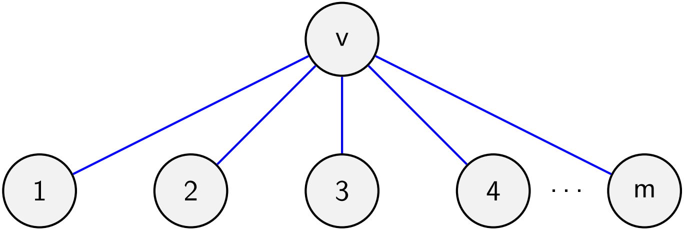

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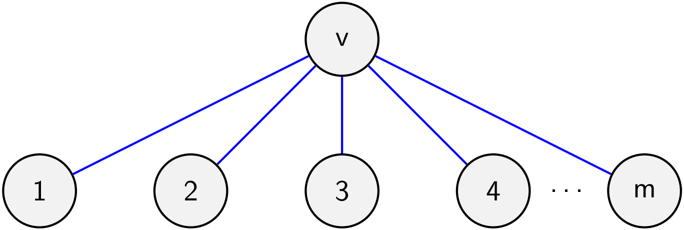

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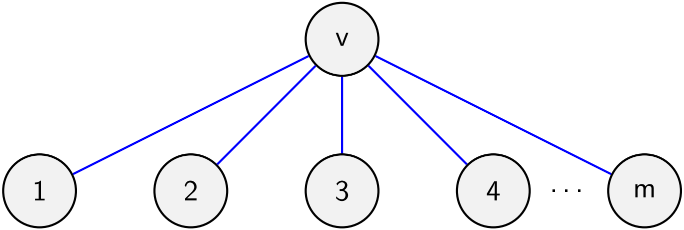

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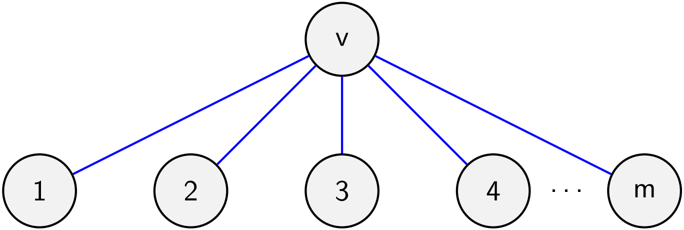

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Case 2.3 Neither. **Impossible** since $m = R(a, b-1)$.

All Verts: Small Red Deg and Small Blue Deg

Case 3 Negate Case 1 and Case 2:

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Hence

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Not possible since every vertex of K_n has degree $n-1$.

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We need a theorem.

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We need a theorem. We first do an example.

A Graph on 9 Vertices with all verts Deg 3?

Thm There is NO graph on 9 verts, with every vertex of deg 3.

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We generalize this on the next slide.

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And NOW to our improvements on small Ramsey numbers.

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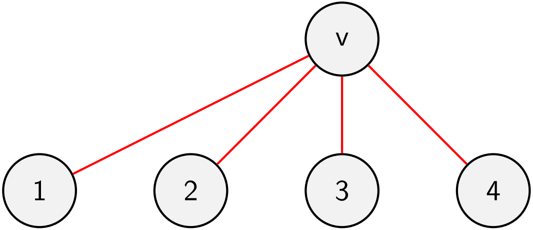

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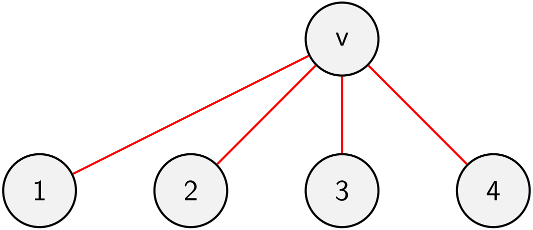

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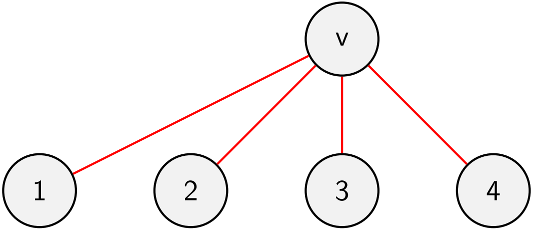

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$R(3,4) \leq 9$ Case 2

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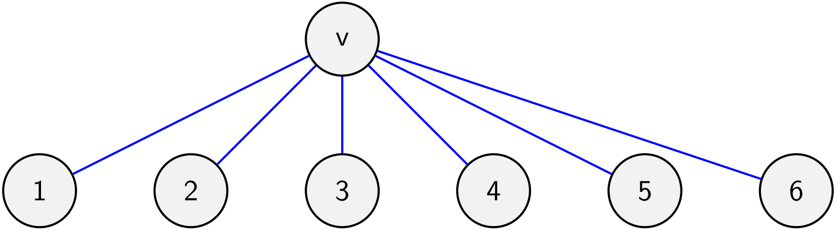

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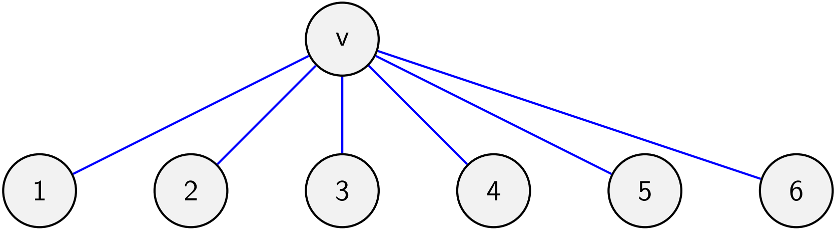

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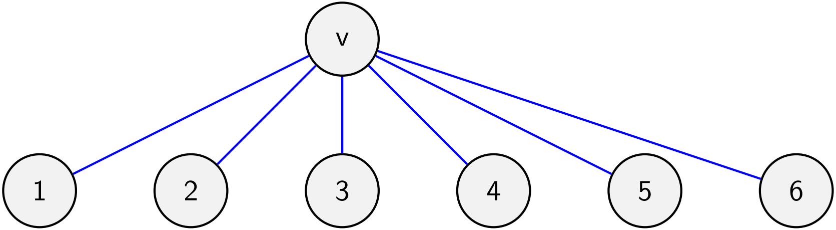

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This is impossible!

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Theorem $R(a,b) \le$

1. $R(a,b-1) + R(a-1,b)$ always.
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Proof left to the Reader.

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Are these tight?

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- $R(3,6) \leq R(2,6) + R(3,5) \leq 6 + 14 - 1 = 19.$
- $R(3,7) \leq R(2,7) + R(3,6) \leq 7 + 19 = 26$
- $R(4,4) \leq R(3,4) + R(4,3) \leq 9 + 9 = 18.$
- $R(4,5) \leq R(3,5) + R(4,4) \leq 14 + 18 - 1 = 31.$
- $R(5,5) \leq R(4,5) + R(5,4) = 62.$

Are these tight? Some yes, some no.

$R(3,3) \geq 6$

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$R(3,3) = 6$ as shown in prior slide.

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Same idea as above for K_5, but more cases for algebra.
UPSHOT $R(4,4) = 18$ and the coloring used math of interest!

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This is a subgraph of the $R(3,5)$ graph

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Can we extend these Patterns?

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THATS IT.

No other $R(a,b)$ are known using NICE methods.

Summary of Bounds

$R(a, b)$	Old Bound	New Bound	Opt	Int?
$R(3,3)$	6	6	6	Y
$R(3,4)$	10	9	9	Y
$R(3,5)$	15	14	14	Y
$R(3,6)$	21	19	18	Lower-Y
$R(3,7)$	28	27	23	Lower-Y
$R(4,4)$	20	18	18	Y
$R(4,5)$	35	31	25	N
$R(5,5)$	70	62	≤ 46	N

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$R(5,5)$: See the assigned paper for more on this.

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   (Joel Spencer) The Law of Small Numbers: Patterns that persist for small numbers will vanish when the calculations get to hard.

2. Seemed like a nice **Math** problem that would involve interesting and perhaps deep mathematics. No. The work on it is interesting and clever, but (1) the math is not deep, and (2) progress is slow.