BILL, RECORD LECTURE!!!!

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An Application of Ramsey's Theorem to Logic

William Gasarch

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We will assume E is symmetric and not reflexive.

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For all $n \geq 2$ there is G with n vertex that satisfies this sentence.

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Definition If ϕ is a sentence in the language of graphs then $\mathbf{spec}(\phi)$ is the set of all n such that there is G on n vertices such that $G \models \phi$.

Spectrum: Examples

$$\phi = (\exists x_1, x_2, x_3)[E(x_1, x_2) \wedge E(x_1, x_3)]$$

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$\mathrm{spec}(\phi) = \{3, 4, 5, \ldots\}$

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$\mathrm{spec}(\phi) = \{0, 2, 4, 6, \ldots, \}$

Spectrum: Another Example

$(\forall x_1, x_2, x_3)$
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$\mathrm{spec}(\phi) = \{2\}$.

Note how Simple Those Spectrum's Were

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Proof Use brute force.

We will use Lemma without comment.

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Note For many (ϕ, G) can do much better than brute force.

Main Theorem

Theorem The following function is computable: Given ϕ, an $\exists^*\forall^*$ sentence in the theory of graphs, output $\mathrm{spec}(\phi)$. ($\mathrm{spec}(\phi)$ will be a finite or cofinite set; hence it will have an easy description.)

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We will take

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Let $G \models \phi$ with witnesses $v_1, \ldots, v_n$. Let H be an induced subgraph of G that contains $v_1, \ldots, v_n$. Then $H \models \phi$.

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Proof of Claim 1 Let $G = (V, E)$ and $H = (V', E')$ where $V' \subseteq V$. Since $G \models \phi$

$$G \models (\forall y_1 \in V) \cdots (\forall y_m \in V)[\psi(v_1, \ldots, v_n, y_1, \ldots, y_m)]$$

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Proof of Claim 1 Let $G = (V,E)$ and $H = (V',E')$ where $V' \subseteq V$. Since $G \models \phi$

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End of Proof of Claim 1

Claim 2, The Main Claim

$$\phi = (\exists x_1)\cdots(\exists x_n)(\forall y_1)\cdots(\forall y_m)[\psi(x_1,\ldots,x_n,y_1,\ldots,y_m)]$$

If $(\exists N \geq n + 2^n R(m))[N \in \mathrm{spec}(\phi)]$ then

$$\{n+m,\ldots,n+2^n R(m),\ldots\} \subseteq \mathrm{spec}(\phi).$$

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Proof of Claim 2

Since $N \in \mathrm{spec}(\phi)$ there exists $G = (V, E)$, a graph on N vertices such that $G \models \phi$. Let $v_1, \ldots, v_n$ be such that:

$$(\forall y_1) \cdots (\forall y_m)[\psi(v_1, \ldots, v_n, y_1, \ldots, y_m)].$$

(Proof continued on next slide)

Proof of Claim 2 Continued

$$(\forall y_1) \cdots (\forall y_m)[\psi(v_1, \ldots, v_n, y_1, \ldots, y_m)].$$

Let $X = \{v_1, \ldots, v_n\}$ and $U = V - X$.

Proof of Claim 2 Continued

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Let $X=\{v_1,\ldots,v_n\}$ and $U=V-X$.
Note that $|U-X|\geq 2^nR(m)$. We use $2^nR(m)$ elements of it which we denote

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Proof of Claim 2 Continued

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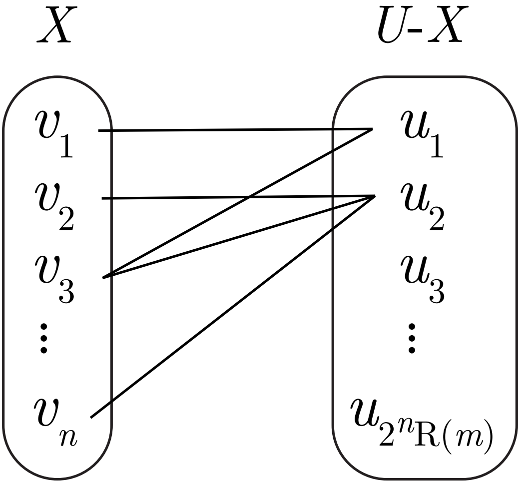
Picture on next slide.

X and $U - X$



Proof of Claim 2 Cont: Pigeonhole

We define a 2^n-Coloring of U. $u \in U$ maps to $(b_1, \ldots, b_n)$:

$$b_i = \begin{cases} 0 \text{ if } (u, v_i) \notin E \\ 1 \text{ if } (u, v_i) \in E \end{cases} \tag{1}$$

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Hence every $u \in U$ is mapped to a description of how it relates to every element in X. Since $|U| \geq 2^n R(m)$ there exists $R(m)$ vertices,

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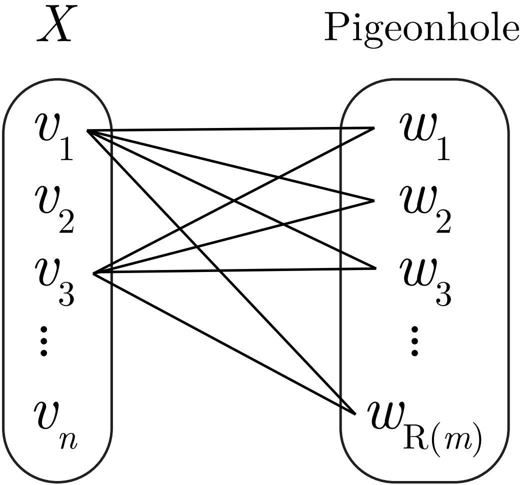
So they all look the same to U. (Picture on the next slide.)

w_i's Look the Same to U



Proof of Claim 2 Cont: Ramsey

Apply Ramsey's Theorem to the graph on

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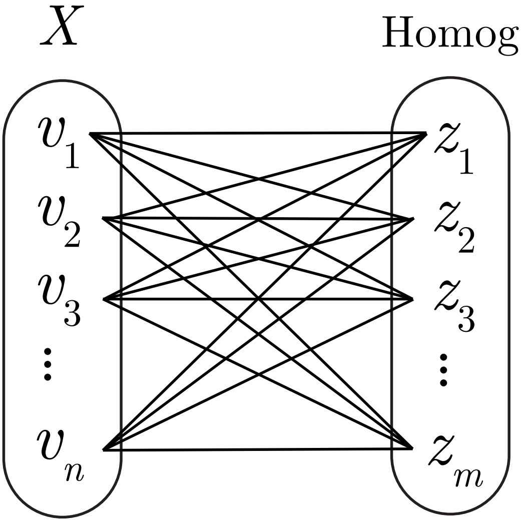
(The case where the form a clique is similar.)
We call the set **Super Homog** since it looks the same to U and to each other.
Picture on the next slide.

The Super Homog Set



Proof of Claim 2 Continued

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Example All z_i have edge to $\{v_1, v_3, v_{17}\}$ but no other v_j.

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Let H_0 be G restricted to $X \cup \{z_1,\ldots,z_m\}$. By Claim 1 $H_0 \models \phi$. For every $p \geq 1$ we form a graph $H_p = (V_p, E_p)$ on $n+m+p$ vertices such that $H_p \models \phi$:

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Informally add $m+p$ vertices that are **just like the z_i's**.

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Formally Next Slide.

Proof of Claim 2 Continued, Formal $H_p = (V_p, E_p)$

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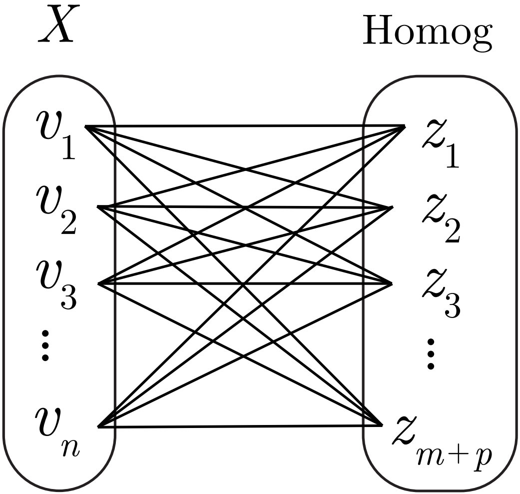
End of Proof of Claim 2

Can Add Vertices



Claim 3

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We put a subcase of Claim 2, and Claim 3, next to each other to recap what we know.
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End of Proof of Main Theorem

Other Sentences. Part I

What other Sentences could we look at?

$\exists^*\forall^*$ sentences with more complicated objects than graphs.

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***a*-ary Hypergraphs** a-ary Hyperedges. c-colored hypergraph. For example every triple is colored R or B or G. Or any finite number of colors.
Key ingredient Ramsey theory on a-hypergraphs.

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Many Predicates We could have $U_1(x)$,..., $U_7(x)$ (all 19-valued). $E_1(x,y)$,...,$E_{13}(x,y)$ (all 4-colored).

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GUESS is 19^7.

Key ingredient Ramsey theory on $\le a$-hypergraphs since use it on Unary THEN binary, THEN (if more arities then more).

Morgan Sentences

$(\exists^*\forall^*)^*$-sentences, only predicate $E(x, y)$. **Morgan** sentences.

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$$\{4, 7, 10, \ldots\} \cup \{11, 22, 33, \ldots\}.$$

Known If A is a Union of AP's then $(\exists\phi)[\mathrm{spec}(\phi) = A]$.

Other Sentences. Part III

$(\exists^*\forall^*)^*$-sentences, predicates of arity $\leq a$-ary. **McKenzie** sentences (that name is real).

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Is spec for McKenzie Sentences decidable? **Vote**.

YES, NO, Unknown to Science.YES.
Known If ϕ is a Mackenzie sentence then $\mathrm{spec}(\phi) \in \mathit{EXPTIME}$.

Other Sentences. Part III

$(\exists^*\forall^*)^*$-sentences, predicates of arity $\leq a$-ary. **McKenzie** sentences (that name is real).

Is spec for McKenzie Sentences decidable? **Vote**.

YES, NO, Unknown to Science.YES.
Known If ϕ is a Mackenzie sentence then $\mathrm{spec}(\phi) \in EXPTIME$.
Also Known If $A \in EXPTIME$ then there exists Mackenzie ϕ such that $\mathrm{spec}(\phi) = A$.

App, “App”, or ““App””

Vote App OR “App” OR ““App””

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““App”” This would be unfair. I reserve the 4-quotes if either NOBODY cares or ONLY I care. (When I prove primes are infinite FROM van Der Waerden's Theorem, feel free to use 4 quotes. I am not kidding.)

Vote App OR “App” OR ““App””