An Exposition of Ramsey's Result in Logic
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1 Introduction

In Ramsey's celebrated paper [4] (see also [2],[3]) his goal was to solve a problem in logic. In this note we discuss what he proved in logic.

We will first state and prove his theorem in logic for undirected graphs (no self loops), and then we will state and prove his theorem in logic for colored hypergraphs.

Def 1.1

- A *graph* is a pair (V, E) where E is a subset of unordered pairs of distinct elements of V. V is referred to as the set of *vertices*. E is referred to as the set of *edges*.
- A *clique* in a graph is a set of vertices such that every pair of vertices in it has an edge.
- An *independent set* in a graph is a set of vertices such that every pair of vertices in it has an edge.

The following is a subcase of Ramsey's Combinatorial theorem.

Theorem 1.2 *For all m there exists a number $R(m)$ such that, for every graph on $R(m)$ vertices, there is either a clique or independent set of size m.*

Note 1.3 It is well known that $2^{m/2} \le R(m) \le 2^{2m}$. A more sophisticated proof, by David Conlon [1] yields, for all k, $n \ge k^{-D\frac{\log k}{\log\log k}}\binom{2k}{k}$ suffices, where D is some constant. A simple probabilistic argument shows that $n \ge (1 + o(1))\frac{1}{e\sqrt{2}})k2^{k/2}$ is necessary. A more sophisticated argument shown by Spencer [5] (see [2]) shows $n \ge (1 + o(1))\frac{\sqrt{2}}{e}k2^{k/2}$ is necessary.

Def 1.4 A sentence is in *the language of graphs* if it only has the usual logical symbols, E a 2-ary predicate, and $=$. We will interpret such sentences as being about undirected graphs with no self loops. Hence we will implicitly assume (1) $E(x, y)$ iff $E(y, x)$ and (2) $\neg E(x, x)$. All of the variables are quanfitied; hence, if G is a graph and ϕ is a sentence, either ϕ is true of G or ϕ is false of G.

Def 1.5 An E^*A^* sentence is one that begins with $\exists$ quantifiers, then has $\forall$ quantifiers, and then has a quantifier-free formula.

Def 1.6 If ϕ is a sentence in the language of graphs then spec(ϕ) is the set of all n such that there is G on n vertices such that $G \models \phi$.

Def 1.7 If ϕ is a sentence in the language of graphs then spec(ϕ) is the set of all n such that there is an undirected graph G with no self-loops on n vertices where $G \models \phi$.-

Convention 1.8 For ease of notation we make the following conventions.

- If there is a contiguous string of the same type of quantifiers then all of the variables in it are distinct. Hence
$$(\exists x_1)(\exists x_2)(\forall y_1)(\forall y_2)[\phi(x_1, x_2, y_1)]$$
actually means
$$(\exists x_1)(\exists x_2 \neq x_1)(\forall y_1)(\forall y_2 \neq y_1)[\phi(x_1, x_2, y_1)]$$
- There are no self-loops. Hence $E(x, y)$ means $E(x, y) \wedge x \neq y$.
- E is symmetric. So $E(x, y)$ means $E(x, y) \wedge E(y, x)$.

Example 1.9

1. $$\phi = (\forall x)(\forall y)[E(x, y)].$$
   This states that every pair of distinct vertices has an edge. For all n $K_n \models \phi$. Hence, $\mathrm{spec}(\phi) = \mathsf{N}$.
2. $$\phi = (\exists x, y, z)(\forall w)[E(w, x) \wedge E(w, y) \wedge E(w, z)].$$
   ϕ states that there are three distinct vertices x, y, z such that every $w \notin \{x, y, z\}$ is connected to x and y and z. For all $n \geq 0$ $K_{n,3} \models \phi$. If G has on 0,1, or 2 vertices then $G \not\models \phi$. Hence, $\mathrm{spec}(\phi) = \{3, 4, 5, \ldots, \}$. (tNote that $K_{0,3} \models \phi$ vacuously.)
3. $$\phi = (\exists x_1)(\exists x_2)(\forall y)[x_1 = y \vee x_2 = y].$$
   If G is a graph on 2 vertices then $G \models \phi$; however, if G has any other number of vertices then $G \not\models \phi$. Hence $spec(\phi) = \{2\}$.

Note that in all three examples $\mathrm{spec}(\phi)$ was either co-finite or finite. We will later see that, for all ϕ, this is the case.

2 Definitions and a Lemma Needed for the Graph case

Lemma 2.1

1. *The following is decidable: Given a sentence ϕ and a graph G, determine if $G \models \phi$.*
2. *The following is decidable: Given a sentence ϕ and a number n, determine if $n \in \mathrm{spec}(\phi)$.*

Proof: Use brute force. ∎

We will use Lemma 2.1 without comment.

3 Ramsey's Theorem in Logic on Graphs

The following is a simple case of what Ramsey proved.

Theorem 3.1 *The following function is computable: Given a* E^*A^* *sentence* ϕ*, output* spec(ϕ)*.* *(*spec(ϕ) *will be a finite or cofinite set; hence it will have an easy description.)*

Proof:
Claim 1: Let $G \models \phi$ with witnesses $v_1, \ldots, v_n$. then any induced subgraph H of G that contains $v_1, \ldots, v_n$ satisfies ϕ.
Proof of Claim 1:
The statement

$$(\forall y_1) \cdots (\forall y_m)[\psi(x_1, \ldots, x_n, y_1, \ldots, y_m)]$$

is true in H since it is true in G and now there are just less cases to check.
End of Proof of Claim 1

Claim 2: Let $N_0 = n + 2^n R(m)$.

1. If there exists $N_0 \geq N$ such that $N \in \text{spec}(\phi)$ then

$$\{n + m, \ldots, N_0, \ldots, \} \subseteq \text{spec}(\phi).$$

   (We show the N_0 for pedagogical value later.)

2. If $N_0 \notin \text{spec}(\phi)$ then

$$\text{spec}(\phi) \subseteq \{0, \ldots, N_0 - 1\}.$$

Proof of Claim 2:
1) Since $N_0 \geq n + 2^n R(m) \in \text{spec}(\phi)$ there exists $G = (V, E)$, a graph on N_0 vertices, where ϕ is true. Let $v_1, \ldots, v_n$ be vertices such that the following is true of G:

$$(\forall y_1) \cdots (\forall y_m)[\psi(v_1, \ldots, v_n, y_1, \ldots, y_m)].$$

Let $X = \{v_1, \ldots, v_n\}$ and $U = V - X$. Note that $|U| \geq 2^n R(m)$. Map every $u \in U$ to $(b_1, \ldots, b_n) \in \{0, 1\}^n$ such that

$$b_i = \begin{cases} 0 \text{ if } (u, v_i) \notin E \\ 1 \text{ if } (u, v_i) \in E \end{cases} \tag{1}$$

Hence every $u \in U$ is mapped to a description of how it relates to every element in X. Since $|U| \geq 2^n R(m)$ there exists $R(m)$ vertices that map to the same vector. Apply Ramsey's theorem to these $R(m)$ vertices to obtain $u_1, \ldots, u_m$ such that the following are true.

- Either the u_i's form a clique or the u_i's form an ind. set. We will assume the u_i's form a clique (the other case is similar).
- All of the u_i's map to the same vector. Hence they all look the same to $v_1, \ldots, v_n$.

Let H_0 be the graph restricted to $X \cup \{u_1, \ldots, u_m\}$. By Claim 1.a H_0 satisfied ϕ. For every $p \geq 1$ we form a graph $H_p = (V_p, E_p)$ on $n + m + p$ vertices such that $H_p \models \phi$.

- $V_p = X \cup \{u_1, \ldots, u_m, u_{m+1}, \ldots, u_{m+p}\}$ where $u_{m+1}, \ldots, u_{m+p}$ are new vertices.
- E_p is the union of the following edges.
  - The edges in H_0,
  - For all $1 \le i < j \le n + m + p$ put an edge between u_i and u_j. (If $i, j \le m$ then there is already an edge there.)
  - Let $(b_1, \ldots, b_n)$ be the vector that all of the elements of $\{u_1, \ldots, u_m\}$ mapped to. For $m + 1 \le j \le m + p$, for $1 \le i \le m$ such that $b_i = 1$, put an edge between u_j and v_i.

As far as X is concerned, all of the $u_1, \ldots, u_{m+p}$ look the same. Hence any subset of the $\{u_1, \ldots, u_{m+p}\}$ of size m will look just like $u_1, \ldots, u_m$ as far as both X is concerned and as far as their connectivity to each other. Hence $H_p \models \phi$. Hence $n + m + p \in \text{spec}(\phi)$.

2) By Part 1 of this Claim,

$$\{N_0, \ldots\} \cap \text{spec}(\phi) \neq \emptyset \implies \{n + m, \ldots, N_0, \ldots\} \subseteq \text{spec}(\phi).$$

We take the contrapositive with a weaker premise.

$$N_0 \notin \text{spec}(\phi) \implies \{N_0, \ldots\} \cap \text{spec}(\phi) = \emptyset$$

Clearly this implies

$$\text{spec}(\phi) \subseteq \{0, \ldots, N_0 - 1\}.$$

End of Proof of Claim 2

We can now give an algorithm for this problem:

1. Input
$$\phi = (\exists x_1) \cdots (\exists x_n)(\forall y_1) \cdots (\forall y_m)[\psi(x_1, \ldots, x_n, y_1, \ldots, y_m)].$$
2. Let $N_0 = n + 2^n R(m)$. Determine if $N_0 \in \text{spec}(\phi)$.

   (a) If YES then by Claim 2a $\{n + m, \ldots\} \subseteq \text{spec}(\phi)$.
   For $0 \le i \le n + m - 1$ test if $i \in \text{spec}(\phi)$. You now know $\text{spec}(\phi)$ which is co-finite. Output it.

   (b) If NO then, by Claim 2b $\text{spec}(\phi) \subseteq \{0, \ldots, N_0 - 1\}$.
   For $n+1 \le i \le N_0$ test if $i \in \text{spec}(\phi)$. You now know $\text{spec}(\phi)$ which is finite set. Output it.

∎

4 What Ramsey Really Did

Ramsey did not work with graphs. He didn't even work with hypergraphs. He worked with colored $\le a$-ary hypergraphs.

We state his main theorem and then discuss the proof.

Def 4.1 Let $a, c \in \mathsf{N}$. A *c-colored $\le a$-ary hypergraph* is:

1. a set of vertices V,
2. a set $E \subseteq \binom{V}{\le a}$,
3. a map of from E to $[c]$.

Theorem 3.1 was about the logic of graphs. The only symbol was $E(x, y)$. To pin down the logic of c-colored $\le a$-ary hypergraphs we need symbols that tell us both arity and color. Hence we have, for $1 \le i \le a$ and $1 \le d \le c$, the i-ary predicate $E_a^i(x_1, \ldots, x_i)$.

We can now state Ramsey's Theorem in Logic.

Theorem 4.2 *The following function is computable: Given ϕ, a E^*A^* sentence in the language of c-colored, $\le$ a-ary hypergraphs of the form output* spec(ϕ)*. (*spec(ϕ) *will be a finite or cofinite set; hence it will have an easy description.)*

The proof of Theorem 4.2 is similar to the proof of Theorem 3.1. The main difference is that we use an iterated hypergraph Ramsey Theorem rather than the version on graphs.

References

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