Rado's Thm

Exposition by William Gasarch

July 24, 2024

Recall Schur's Thm

Thm $(\forall c)(\exists S = S(c))$ st $\forall$ $\mathrm{COL} : [S] \rightarrow [c]$ $\exists x, y, z$ st

Recall Schur's Thm

Thm $(\forall c)(\exists S = S(c))$ st $\forall$ $\mathrm{COL} : [S] \rightarrow [c]$ $\exists x, y, z$ st

- $\mathrm{COL}(x) = \mathrm{COL}(y) = \mathrm{COL}(z)$

Recall Schur's Thm

Thm $(\forall c)(\exists S = S(c))$ st $\forall$ $\mathrm{COL} : [S] \rightarrow [c]$ $\exists x, y, z$ st

- $\mathrm{COL}(x) = \mathrm{COL}(y) = \mathrm{COL}(z)$
- $x + y = z$

Recall Schur's Thm

Thm $(\forall c)(\exists S = S(c))$ st $\forall$ $\mathrm{COL} : [S] \rightarrow [c]$ $\exists x, y, z$ st

- $\mathrm{COL}(x) = \mathrm{COL}(y) = \mathrm{COL}(z)$
- $x + y = z$

We proved using Ramsey's Thm.

Recall Schur's Thm

Thm $(\forall c)(\exists S = S(c))$ st $\forall$ $\mathrm{COL} : [S] \rightarrow [c]$ $\exists x, y, z$ st

- $\mathrm{COL}(x) = \mathrm{COL}(y) = \mathrm{COL}(z)$
- $x + y = z$

We proved using Ramsey's Thm.

What about other equations?

Terminology: Mono Solution

Def Let $E(x_1, \ldots, x_n)$ be an equation (e.g., $x + y = z$).

Terminology: Mono Solution

Def Let $E(x_1, \ldots, x_n)$ be an equation (e.g., $x + y = z$).
Let $R, c \in \mathbb{N}$.

Terminology: Mono Solution

Def Let $E(x_1, \ldots, x_n)$ be an equation (e.g., $x + y = z$).
Let $R, c \in \mathbb{N}$.
Let $\mathrm{COL}\colon [R] \rightarrow [c]$.

Terminology: Mono Solution

Def Let $E(x_1, \ldots, x_n)$ be an equation (e.g., $x + y = z$).
Let $R, c \in \mathbb{N}$.
Let $\mathrm{COL}\colon [R] \rightarrow [c]$.
A **monochromatic solution (mono sol)** is a tuple of numbers in $[R]$, $(d_1, \ldots, d_n)$ such that

Terminology: Mono Solution

Def Let $E(x_1, \ldots, x_n)$ be an equation (e.g., $x + y = z$).
Let $R, c \in \mathbb{N}$.
Let $\mathrm{COL}\colon [R] \rightarrow [c]$.
A **monochromatic solution (mono sol)** is a tuple of numbers in $[R]$, $(d_1, \ldots, d_n)$ such that

1) $d_1, \ldots, d_n$ are all the same color.

Terminology: Mono Solution

Def Let $E(x_1, \ldots, x_n)$ be an equation (e.g., $x + y = z$).
Let $R, c \in \mathbb{N}$.
Let $\mathrm{COL}\colon [R] \rightarrow [c]$.
A **monochromatic solution (mono sol)** is a tuple of numbers in $[R]$, $(d_1, \ldots, d_n)$ such that

1) $d_1, \ldots, d_n$ are all the same color.

2) $E(d_1, \ldots, d_n)$ is true.

Terminology: Mono Solution

Def Let $E(x_1, \ldots, x_n)$ be an equation (e.g., $x + y = z$).
Let $R, c \in \mathbb{N}$.
Let $\mathrm{COL}\colon [R] \to [c]$.
A **monochromatic solution (mono sol)** is a tuple of numbers in $[R]$, $(d_1, \ldots, d_n)$ such that

1) $d_1, \ldots, d_n$ are all the same color.

2) $E(d_1, \ldots, d_n)$ is true.

A **distinct monochromatic solution (d-mono sol)** is a mono sol where all of the elements are different.

Terminology: Mono Solution

Def Let $E(x_1, \ldots, x_n)$ be an equation (e.g., $x + y = z$).
Let $R, c \in \mathbb{N}$.
Let $\mathrm{COL}\colon [R] \rightarrow [c]$.
A **monochromatic solution (mono sol)** is a tuple of numbers in $[R]$, $(d_1, \ldots, d_n)$ such that

1) $d_1, \ldots, d_n$ are all the same color.

2) $E(d_1, \ldots, d_n)$ is true.

A **distinct monochromatic solution (d-mono sol)** is a mono sol where all of the elements are different.

We can restate Schur's Thm

Terminology: Mono Solution

Def Let $E(x_1, \ldots, x_n)$ be an equation (e.g., $x + y = z$).
Let $R, c \in \mathbb{N}$.
Let $\mathrm{COL}\colon [R] \rightarrow [c]$.
A **monochromatic solution (mono sol)** is a tuple of numbers in $[R]$, $(d_1, \ldots, d_n)$ such that

1) $d_1, \ldots, d_n$ are all the same color.

2) $E(d_1, \ldots, d_n)$ is true.

A **distinct monochromatic solution (d-mono sol)** is a mono sol where all of the elements are different.

We can restate Schur's Thm
Thm $(\forall c)(\exists S = S(c))$ st $\forall\ \mathrm{COL} : [S] \rightarrow [c]$ there is a mono sol to $x + y = z$.

Terminology: Mono Solution

Def Let $E(x_1, \ldots, x_n)$ be an equation (e.g., $x + y = z$).
Let $R, c \in \mathbb{N}$.
Let $\mathrm{COL}\colon [R] \to [c]$.
A **monochromatic solution (mono sol)** is a tuple of numbers in $[R]$, $(d_1, \ldots, d_n)$ such that

1) $d_1, \ldots, d_n$ are all the same color.

2) $E(d_1, \ldots, d_n)$ is true.

A **distinct monochromatic solution (d-mono sol)** is a mono sol where all of the elements are different.

We can restate Schur's Thm
Thm $(\forall c)(\exists S = S(c))$ st $\forall$ $\mathrm{COL} : [S] \to [c]$ there is a mono sol to $x + y = z$.

(We can modify the proof to get a d-mono sol.)

Terminology: Regular

Def Let $E(x_1, \ldots, x_n)$ be an equation (e.g., $x + y = z$).
E is **regular** if the following is true:

Terminology: Regular

Def Let $E(x_1, \ldots, x_n)$ be an equation (e.g., $x + y = z$).
E is **regular** if the following is true:

$(\forall c \in \mathbb{N})(\exists R \in \mathbb{N})\ \forall\ \mathrm{COL}\colon [R] \rightarrow [c]$ there is a mono sol.

Terminology: Regular

Def Let $E(x_1, \ldots, x_n)$ be an equation (e.g., $x + y = z$).
E is **regular** if the following is true:

$(\forall c \in \mathbb{N})(\exists R \in \mathbb{N})\ \forall\ \mathrm{COL}\colon [R] \rightarrow [c]$ there is a mono sol.

One can define d-regular with d-mono sol.

Terminology: Regular

Def Let $E(x_1, \ldots, x_n)$ be an equation (e.g., $x + y = z$).
E is **regular** if the following is true:

$(\forall c \in \mathbb{N})(\exists R \in \mathbb{N})\ \forall\ \mathrm{COL}\colon [R] \rightarrow [c]$ there is a mono sol.

One can define d-regular with d-mono sol.

We can restate Schur's Thm

Terminology: Regular

Def Let $E(x_1, \ldots, x_n)$ be an equation (e.g., $x + y = z$).
E is **regular** if the following is true:

$(\forall c \in \mathbb{N})(\exists R \in \mathbb{N}) \ \forall \ \mathrm{COL}\colon [R] \to [c]$ there is a mono sol.

One can define d-regular with d-mono sol.

We can restate Schur's Thm
Thm $x + y = z$ is regular. (Can also show d-regular.)

$2w + 3x = 5y$

Thm $2w + 3x = 5y$ is regular.

$2w + 3x = 5y$

Thm $2w + 3x = 5y$ is regular.

This is a stupid thm.

$2w + 3x = 5y$

Thm $2w + 3x = 5y$ is regular.

This is a stupid thm.

Take $x = y = z = 1$. Or any $x = y = z$.

$2w + 3x = 5y$

Thm $2w + 3x = 5y$ is d-regular.

$2w + 3x = 5y$

Thm $2w + 3x = 5y$ is d-regular.

Let $c \in \mathbb{N}$. Use VDW's thm with c and with k we pick later.

$2w + 3x = 5y$

Thm $2w + 3x = 5y$ is d-regular.

Let $c \in \mathbb{N}$. Use VDW's thm with c and with k we pick later.
$\exists W$ for all $\mathrm{COL}[W] \rightarrow [c]$ $\exists a, d$

$2w + 3x = 5y$

Thm $2w + 3x = 5y$ is d-regular.

Let $c \in \mathbb{N}$. Use VDW's thm with c and with k we pick later.
$\exists W$ for all $\mathrm{COL}[W] \rightarrow [c]$ $\exists a, d$

$$a, a+d, \ldots, a+(k-1)d \text{ are all the same color}$$

$2w + 3x = 5y$

Thm $2w + 3x = 5y$ is d-regular.

Let $c \in \mathbb{N}$. Use VDW's thm with c and with k we pick later.
$\exists W$ for all $\mathrm{COL}[W] \rightarrow [c]$ $\exists a, d$

$$a, a + d, \ldots, a + (k - 1)d \text{ are all the same color}$$

We pick $0 \le W, X, Y \le k$ distinct later and then set

$2w + 3x = 5y$

Thm $2w + 3x = 5y$ is d-regular.

Let $c \in \mathbb{N}$. Use VDW's thm with c and with k we pick later.
$\exists W$ for all $\mathrm{COL}[W] \rightarrow [c]$ $\exists a, d$

$$a, a + d, \dots, a + (k - 1)d \text{ are all the same color}$$

We pick $0 \le W, X, Y \le k$ distinct later and then set
$w = a + Wd$

$2w + 3x = 5y$

Thm $2w + 3x = 5y$ is d-regular.

Let $c \in \mathbb{N}$. Use VDW's thm with c and with k we pick later.
$\exists W$ for all $\mathrm{COL}[W]\rightarrow[c]$ $\exists a, d$

$$a, a+d, \ldots, a+(k-1)d \text{ are all the same color}$$

We pick $0 \le W, X, Y \le k$ distinct later and then set
$w = a + Wd$ $\qquad$ $x = a + Xd$

$2w + 3x = 5y$

Thm $2w + 3x = 5y$ is d-regular.

Let $c \in \mathbb{N}$. Use VDW's thm with c and with k we pick later.
$\exists W$ for all $\mathrm{COL}[W] \rightarrow [c]$ $\exists a, d$

$$a, a+d, \ldots, a+(k-1)d \text{ are all the same color}$$

We pick $0 \le W, X, Y \le k$ distinct later and then set
$w = a + Wd \qquad x = a + Xd \qquad y = a + Yd$

$2w + 3x = 5y$

Thm $2w + 3x = 5y$ is d-regular.

Let $c \in \mathbb{N}$. Use VDW's thm with c and with k we pick later.
$\exists W$ for all $\mathrm{COL}[W] \rightarrow [c]$ $\exists a, d$

$$a, a + d, \ldots, a + (k - 1)d \text{ are all the same color}$$

We pick $0 \le W, X, Y \le k$ distinct later and then set
$w = a + Wd \qquad x = a + Xd \qquad y = a + Yd$
Good News: $\mathrm{COL}(w) = \mathrm{COL}(x) = \mathrm{COL}(y)$.

$2w + 3x = 5y$

Thm $2w + 3x = 5y$ is d-regular.

Let $c \in \mathbb{N}$. Use VDW's thm with c and with k we pick later.
$\exists W$ for all $\mathrm{COL}[W] \rightarrow [c]$ $\exists a, d$

$$a, a+d, \ldots, a+(k-1)d \text{ are all the same color}$$

We pick $0 \le W, X, Y \le k$ distinct later and then set
$w = a + Wd$ $\quad$ $x = a + Xd$ $\quad$ $y = a + Yd$
Good News: $\mathrm{COL}(w) = \mathrm{COL}(x) = \mathrm{COL}(y)$.
Want

$2w + 3x = 5y$

Thm $2w + 3x = 5y$ is d-regular.

Let $c \in \mathbb{N}$. Use VDW's thm with c and with k we pick later.
$\exists W$ for all $\mathrm{COL}[W] \rightarrow [c]$ $\exists a, d$

$$a, a+d, \ldots, a+(k-1)d \text{ are all the same color}$$

We pick $0 \le W, X, Y \le k$ distinct later and then set
$w = a + Wd$ $\qquad$ $x = a + Xd$ $\qquad$ $y = a + Yd$
Good News: $\mathrm{COL}(w) = \mathrm{COL}(x) = \mathrm{COL}(y)$.
Want

$$2w + 3x = 5y$$

$2w + 3x = 5y$

Thm $2w + 3x = 5y$ is d-regular.

Let $c \in \mathbb{N}$. Use VDW's thm with c and with k we pick later.
$\exists W$ for all $\mathrm{COL}[W] \rightarrow [c]$ $\exists a, d$

$$a, a+d, \ldots, a+(k-1)d \text{ are all the same color}$$

We pick $0 \le W, X, Y \le k$ distinct later and then set
$w = a + Wd \qquad x = a + Xd \qquad y = a + Yd$
Good News: $\mathrm{COL}(w) = \mathrm{COL}(x) = \mathrm{COL}(y)$.
Want

$$2w + 3x = 5y$$

$$2(a + Wd) + 3(a + Xd) = 5(a + Yd)$$

How We Get What We Want

How We Get What We Want

Want

$2w + 3x = 5y$

How We Get What We Want

Want

$2w + 3x = 5y$

$2(a + Wd) + 3(a + Xd) = 5(a + Yd)$

How We Get What We Want

Want

$2w + 3x = 5y$

$2(a + Wd) + 3(a + Xd) = 5(a + Yd)$

$2a + 2Wd + 3a + 3Xd = 5a + 5Y$

How We Get What We Want

Want

$2w + 3x = 5y$

$2(a + Wd) + 3(a + Xd) = 5(a + Yd)$

$2a + 2Wd + 3a + 3Xd = 5a + 5Y$ **WOW** all of the a's Drop out!

How We Get What We Want

Want

$2w + 3x = 5y$

$2(a + Wd) + 3(a + Xd) = 5(a + Yd)$

$2a + 2Wd + 3a + 3Xd = 5a + 5Y$ **WOW** all of the a's Drop out!

$2Wd + 3Xd = 5Yd$

How We Get What We Want

Want

$2w + 3x = 5y$

$2(a + Wd) + 3(a + Xd) = 5(a + Yd)$

$2a + 2Wd + 3a + 3Xd = 5a + 5Y$ **WOW** all of the a's Drop out!

$2Wd + 3Xd = 5Yd$ **WOW** all of the d's Drop out!

How We Get What We Want

Want

$2w + 3x = 5y$

$2(a + Wd) + 3(a + Xd) = 5(a + Yd)$

$2a + 2Wd + 3a + 3Xd = 5a + 5Y$ **WOW** all of the a's Drop out!

$2Wd + 3Xd = 5Yd$ **WOW** all of the d's Drop out!

$2W + 3X = 5Y$

How We Get What We Want

Want

$2w + 3x = 5y$

$2(a + Wd) + 3(a + Xd) = 5(a + Yd)$

$2a + 2Wd + 3a + 3Xd = 5a + 5Y$ **WOW** all of the a's Drop out!

$2Wd + 3Xd = 5Yd$ **WOW** all of the d's Drop out!

$2W + 3X = 5Y$

Could do $W = 1,\ X = 1,\ Y = 1$.

How We Get What We Want

Want

$2w + 3x = 5y$

$2(a + Wd) + 3(a + Xd) = 5(a + Yd)$

$2a + 2Wd + 3a + 3Xd = 5a + 5Y$ **WOW** all of the a's Drop out!

$2Wd + 3Xd = 5Yd$ **WOW** all of the d's Drop out!

$2W + 3X = 5Y$

Could do $W = 1,\ X = 1,\ Y = 1.$ But this causes $x = y = z$.

How We Get What We Want

Want

$2w + 3x = 5y$

$2(a + Wd) + 3(a + Xd) = 5(a + Yd)$

$2a + 2Wd + 3a + 3Xd = 5a + 5Y$ **WOW** all of the a's Drop out!

$2Wd + 3Xd = 5Yd$ **WOW** all of the d's Drop out!

$2W + 3X = 5Y$

Could do $W = 1,\ X = 1,\ Y = 1$. But this causes $x = y = z$.

Will do $W = 0,\ X = 5,\ Y = 3$.

How We Get What We Want

Want

$2w + 3x = 5y$

$2(a + Wd) + 3(a + Xd) = 5(a + Yd)$

$2a + 2Wd + 3a + 3Xd = 5a + 5Y$ **WOW** all of the a's Drop out!

$2Wd + 3Xd = 5Yd$ **WOW** all of the d's Drop out!

$2W + 3X = 5Y$

Could do $W = 1,\ X = 1,\ Y = 1$. But this causes $x = y = z$.

Will do $W = 0,\ X = 5,\ Y = 3$.

So get $x = a \qquad x = a + 5d \qquad y = a + 3d$.

Recap The Proof

Thm $2w + 3x = 5y$ is d-regular.

Recap The Proof

Thm $2w + 3x = 5y$ is d-regular.

Given c, let $R = R(c) = W(6, c)$.

Recap The Proof

Thm $2w + 3x = 5y$ is d-regular.

Given c, let $R = R(c) = W(6, c)$.

$\mathrm{COL} : [R] \rightarrow [c]$.

Recap The Proof

Thm $2w + 3x = 5y$ is d-regular.

Given c, let $R = R(c) = W(6, c)$.

$\mathrm{COL} : [R] \rightarrow [c]$.
By VDW $\exists a, d$, $\mathrm{COL}(a) = \mathrm{COL}(a + d) = \cdots = \mathrm{COL}(a + 5d)$.

Recap The Proof

Thm $2w + 3x = 5y$ is d-regular.

Given c, let $R = R(c) = W(6, c)$.

$\mathrm{COL} : [R] \rightarrow [c]$.
By VDW $\exists a, d$, $\mathrm{COL}(a) = \mathrm{COL}(a + d) = \cdots = \mathrm{COL}(a + 5d)$.

$$w = a \qquad x = a + 5d \qquad y = a + 3d$$

Recap The Proof

Thm $2w + 3x = 5y$ is d-regular.

Given c, let $R = R(c) = W(6, c)$.

$\mathrm{COL} : [R] \rightarrow [c]$.
By VDW $\exists a, d$, $\mathrm{COL}(a) = \mathrm{COL}(a+d) = \cdots = \mathrm{COL}(a+5d)$.

$$w = a \qquad x = a + 5d \qquad y = a + 3d$$

$\mathrm{COL}(a+d) = \mathrm{COL}(a+5d) = \mathrm{COL}(a+3d)$

Recap The Proof

Thm $2w + 3x = 5y$ is d-regular.

Given c, let $R = R(c) = W(6, c)$.

$\mathrm{COL} : [R] \rightarrow [c]$.
By VDW $\exists a, d$, $\mathrm{COL}(a) = \mathrm{COL}(a + d) = \cdots = \mathrm{COL}(a + 5d)$.

$$w = a \qquad x = a + 5d \qquad y = a + 3d$$

$\mathrm{COL}(a + d) = \mathrm{COL}(a + 5d) = \mathrm{COL}(a + 3d)$

$2(a) + 3(a + 5d) = 5(a + 3d)$

Recap The Proof

Thm $2w + 3x = 5y$ is d-regular.

Given c, let $R = R(c) = W(6, c)$.

$\mathrm{COL} : [R] \rightarrow [c]$.
By VDW $\exists a, d$, $\mathrm{COL}(a) = \mathrm{COL}(a + d) = \cdots = \mathrm{COL}(a + 5d)$.

$$w = a \qquad x = a + 5d \qquad y = a + 3d$$

$\mathrm{COL}(a + d) = \mathrm{COL}(a + 5d) = \mathrm{COL}(a + 3d)$

$2(a) + 3(a + 5d) = 5(a + 3d)$
Done!

What Was Special About $2w + 3x = 5y$?

$2w + 3x = 5y$

What Was Special About $2w + 3x = 5y$?

$2w + 3x = 5y$

Set $w = a + Wd$, $x = a + Xd$, $y = a + Yd$ and the a's dropped out.

What Was Special About $2w + 3x = 5y$?

$2w + 3x = 5y$

Set $w = a + Wd$, $x = a + Xd$, $y = a + Yd$ and the a's dropped out.

Then all the d's dropped out so we go equation in just W, X, Y.

What Was Special About $2w + 3x = 5y$?

$2w + 3x = 5y$

Set $w = a + Wd$, $x = a + Xd$, $y = a + Yd$ and the a's dropped out.

Then all the d's dropped out so we go equation in just W, X, Y.

What is it about

$$2w + 3x = 5y$$

that made all of the a's drop out? Discuss.

Mini Rado Thm (Statement)

The key to $2w + 3x = 5y$ is that $2 + 3 = 5$.

Mini Rado Thm (Statement)

The key to $2w + 3x = 5y$ is that $2 + 3 = 5$.
Can phrase as $2w + 3x - 5y = 0$ and say sum of coefficients is 0.

Mini Rado Thm (Statement)

The key to $2w + 3x = 5y$ is that $2 + 3 = 5$.
Can phrase as $2w + 3x - 5y = 0$ and say sum of coefficients is 0.

Thm Let $a_1, \ldots, a_m \in \mathbb{N}$ and $b_1, \ldots, b_n \in \mathbb{N}$ be st $\sum_{i=1}^{m} a_i = \sum_{i=1}^{n} b_i$. Then

Mini Rado Thm (Statement)

The key to $2w + 3x = 5y$ is that $2 + 3 = 5$.
Can phrase as $2w + 3x - 5y = 0$ and say sum of coefficients is 0.

Thm Let $a_1, \ldots, a_m \in \mathbb{N}$ and $b_1, \ldots, b_n \in \mathbb{N}$ be st
$\sum_{i=1}^{m} a_i = \sum_{i=1}^{n} b_i$. Then
$\sum_{i=1}^{m} a_i x_i = \sum_{i=1}^{n} b_i y_i$ is d-regular.
(One exception: $x = y$.)

Mini Rado Thm (Statement)

The key to $2w + 3x = 5y$ is that $2 + 3 = 5$.
Can phrase as $2w + 3x - 5y = 0$ and say sum of coefficients is 0.

Thm Let $a_1, \ldots, a_m \in \mathbb{N}$ and $b_1, \ldots, b_n \in \mathbb{N}$ be st $\sum_{i=1}^{m} a_i = \sum_{i=1}^{n} b_i$. Then $\sum_{i=1}^{m} a_i x_i = \sum_{i=1}^{n} b_i y_i$ is d-regular.
(One exception: $x = y$.)

We won't prove this but you have seen most of the ideas needed to prove it.

Other Equations

$2w + 3x = 5y + z$

Thm $2w + 3x = 5y + z$ is d-regular.

$2w + 3x = 5y + z$

Thm $2w + 3x = 5y + z$ is d-regular.
Let $c \in \mathbb{N}$.Use VDW's thm with c and with k we pick later.

$2w + 3x = 5y + z$

Thm $2w + 3x = 5y + z$ is d-regular.
Let $c \in \mathbb{N}$.Use VDW's thm with c and with k we pick later.
$\exists W$ for all $\mathrm{COL}[W] \rightarrow [c]$ $\exists a, d$

$$a, a + d, \ldots, a + (k - 1)d \text{ are all the same color}$$

$2w + 3x = 5y + z$

Thm $2w + 3x = 5y + z$ is d-regular.
Let $c \in \mathbb{N}$.Use VDW's thm with c and with k we pick later.
$\exists W$ for all $\mathrm{COL}[W] \rightarrow [c]$ $\exists a, d$

$$a, a+d, \ldots, a+(k-1)d \text{ are all the same color}$$

We pick $0 \le W, X, Y, Z \le k$ later and then set

$2w + 3x = 5y + z$

Thm $2w + 3x = 5y + z$ is d-regular.
Let $c \in \mathbb{N}$.Use VDW's thm with c and with k we pick later.
$\exists W$ for all $\mathrm{COL}[W] \rightarrow [c]$ $\exists a, d$

$$a, a+d, \ldots, a+(k-1)d \text{ are all the same color}$$

We pick $0 \le W, X, Y, Z \le k$ later and then set
$w = a + Wd$

$2w + 3x = 5y + z$

Thm $2w + 3x = 5y + z$ is d-regular.
Let $c \in \mathbb{N}$.Use VDW's thm with c and with k we pick later.
$\exists W$ for all $\mathrm{COL}[W] \rightarrow [c]$ $\exists a, d$

$$a, a+d, \ldots, a+(k-1)d \text{ are all the same color}$$

We pick $0 \le W, X, Y, Z \le k$ later and then set
$w = a + Wd \qquad x = a + Xd$

$2w + 3x = 5y + z$

Thm $2w + 3x = 5y + z$ is d-regular.
Let $c \in \mathbb{N}$.Use VDW's thm with c and with k we pick later.
$\exists W$ for all $\mathrm{COL}[W] \to [c]$ $\exists a, d$

$$a, a+d, \ldots, a+(k-1)d \text{ are all the same color}$$

We pick $0 \le W, X, Y, Z \le k$ later and then set
$w = a + Wd \qquad x = a + Xd \qquad y = a + Yd$

$2w + 3x = 5y + z$

Thm $2w + 3x = 5y + z$ is d-regular.
Let $c \in \mathbb{N}$.Use VDW's thm with c and with k we pick later.
$\exists W$ for all $\mathrm{COL}[W] \rightarrow [c]$ $\exists a, d$

$$a, a+d, \ldots, a+(k-1)d \text{ are all the same color}$$

We pick $0 \le W, X, Y, Z \le k$ later and then set
$w = a + Wd \qquad x = a + Xd \qquad y = a + Yd \qquad z = a + Zd$

$2w + 3x = 5y + z$

Thm $2w + 3x = 5y + z$ is d-regular.
Let $c \in \mathbb{N}$.Use VDW's thm with c and with k we pick later.
$\exists W$ for all $\mathrm{COL}[W]\to[c]$ $\exists a, d$

$$a, a+d, \ldots, a+(k-1)d \text{ are all the same color}$$

We pick $0 \le W, X, Y, Z \le k$ later and then set
$w = a + Wd \qquad x = a + Xd \qquad y = a + Yd \qquad z = a + Zd$
Good News: $\mathrm{COL}(w) = \mathrm{COL}(x) = \mathrm{COL}(y) = \mathrm{COL}(z)$.

$2w + 3x = 5y + z$

Thm $2w + 3x = 5y + z$ is d-regular.
Let $c \in \mathbb{N}$.Use VDW's thm with c and with k we pick later.
$\exists W$ for all $\mathrm{COL}[W] \rightarrow [c]$ $\exists a, d$

$$a, a+d, \ldots, a+(k-1)d \text{ are all the same color}$$

We pick $0 \le W, X, Y, Z \le k$ later and then set
$w = a + Wd \qquad x = a + Xd \qquad y = a + Yd \qquad z = a + Zd$
Good News: $\mathrm{COL}(w) = \mathrm{COL}(x) = \mathrm{COL}(y) = \mathrm{COL}(z)$.
Want

$2w + 3x = 5y + z$

Thm $2w + 3x = 5y + z$ is d-regular.
Let $c \in \mathbb{N}$.Use VDW's thm with c and with k we pick later.
$\exists W$ for all $\mathrm{COL}[W] \rightarrow [c]$ $\exists a, d$

$$a, a + d, \ldots, a + (k-1)d \text{ are all the same color}$$

We pick $0 \le W, X, Y, Z \le k$ later and then set
$w = a + Wd \qquad x = a + Xd \qquad y = a + Yd \qquad z = a + Zd$
Good News: $\mathrm{COL}(w) = \mathrm{COL}(x) = \mathrm{COL}(y) = \mathrm{COL}(z)$.
Want

$$2w + 3x = 5y + z$$

$2w + 3x = 5y + z$

Thm $2w + 3x = 5y + z$ is d-regular.
Let $c \in \mathbb{N}$.Use VDW's thm with c and with k we pick later.
$\exists W$ for all $\mathrm{COL}[W]\to[c]$ $\exists a, d$

$$a, a + d, \ldots, a + (k-1)d \text{ are all the same color}$$

We pick $0 \le W, X, Y, Z \le k$ later and then set
$w = a + Wd \qquad x = a + Xd \qquad y = a + Yd \qquad z = a + Zd$
Good News: $\mathrm{COL}(w) = \mathrm{COL}(x) = \mathrm{COL}(y) = \mathrm{COL}(z)$.
Want

$$2w + 3x = 5y + z$$

$$2(a + Wd) + 3(a + Xd) = 5(a + Yd) + (a + Zd)$$

How We Get What We Want

How We Get What We Want

Want

$2w + 3x = 5y + z$

How We Get What We Want

Want

$2w + 3x = 5y + z$

$2(a + Wd) + 3(a + Xd) = 5(a + Yd) + (a + Zd)$

How We Get What We Want

Want

$2w + 3x = 5y + z$

$2(a + Wd) + 3(a + Xd) = 5(a + Yd) + (a + Zd)$

$$2a + 2Wd + 3a + 3Xd = 5a + 5Y + a + Zd$$

How We Get What We Want

Want

$2w + 3x = 5y + z$

$2(a + Wd) + 3(a + Xd) = 5(a + Yd) + (a + Zd)$

$$2a + 2Wd + 3a + 3Xd = 5a + 5Y + a + Zd$$

WOW

How We Get What We Want

Want

$2w + 3x = 5y + z$

$2(a + Wd) + 3(a + Xd) = 5(a + Yd) + (a + Zd)$

$$2a + 2Wd + 3a + 3Xd = 5a + 5Y + a + Zd$$

WOW Nothing drops out.

How We Get What We Want

Want

$2w + 3x = 5y + z$

$2(a + Wd) + 3(a + Xd) = 5(a + Yd) + (a + Zd)$

$$2a + 2Wd + 3a + 3Xd = 5a + 5Y + a + Zd$$

WOW Nothing drops out.

What to do? Discuss.

How We Get What We Want

Want

$2w + 3x = 5y + z$

$2(a + Wd) + 3(a + Xd) = 5(a + Yd) + (a + Zd)$

$$2a + 2Wd + 3a + 3Xd = 5a + 5Y + a + Zd$$

WOW Nothing drops out.
What to do? Discuss.

We would like to set $z = Zd$ instead of $z = a + Zd$.

How We Get What We Want

Want

$2w + 3x = 5y + z$

$2(a + Wd) + 3(a + Xd) = 5(a + Yd) + (a + Zd)$

$$2a + 2Wd + 3a + 3Xd = 5a + 5Y + a + Zd$$

WOW Nothing drops out.

What to do? Discuss.

We would like to set $z = Zd$ instead of $z = a + Zd$.

Need a Variant of VDW's Thm.

Extended VDW Thm

Extended VDW's Thm

VDW's Thm $(\forall k, c)(\exists W = W(k,c)$ st $\forall\ \mathrm{COL}\colon [W] \rightarrow [c]\ \exists a, d$ st

$$\mathrm{COL}(a) = \cdots = \mathrm{COL}(a + (k-1))d$$

Extended VDW's Thm

VDW's Thm $(\forall k, c)(\exists W = W(k, c)$ st $\forall$ COL: $[W] \rightarrow [c]$ $\exists a, d$ st

$$\mathrm{COL}(a) = \cdots = \mathrm{COL}(a + (k-1))d$$

What about d itself? Can it be the same colors as $a, a+d, \ldots, a+(k-1)d$?

Extended VDW's Thm

VDW's Thm $(\forall k, c)(\exists W = W(k,c)$ st $\forall$ $\mathrm{COL}\colon [W] \rightarrow [c]$ $\exists a, d$ st

$$\mathrm{COL}(a) = \cdots = \mathrm{COL}(a + (k-1))d$$

What about d itself? Can it be the same colors as $a, a+d, \ldots, a+(k-1)d$?

Extended VDW's Thm
EVDW Thm $(\forall k, c)(\exists E = E(k,c)$ st $\forall$ $\mathrm{COL}\colon [E] \rightarrow [c]$ $\exists a, d$ st

$$\mathrm{COL}(a) = \cdots = \mathrm{COL}(a + (k-1)d) = \mathrm{COL}(d)$$

Proof of Extended VDW Thm

Pf. Ind on c.

Proof of Extended VDW Thm

Pf. Ind on c.
$E(k,1) = k$.

Proof of Extended VDW Thm

Pf. Ind on c.
$E(k, 1) = k$.
We show $E(k, c) \leq W(kX, c)$ for a **large** X.

Proof of Extended VDW Thm

Pf. Ind on c.
$E(k,1) = k$.
We show $E(k,c) \le W(kX,c)$ for a **large** X.
$\mathrm{COL}\colon [W(kX,c)] \to [c]$.

Proof of Extended VDW Thm

Pf. Ind on c.
$E(k,1) = k$.
We show $E(k,c) \le W(kX,c)$ for a **large** X.
$\mathrm{COL}\colon [W(kX,c)] \rightarrow [c]$.
By VDW there exists A, D:
$A, A+D, \ldots, A+kXD$ is color (we can assume) c.

Proof of Extended VDW Thm

Pf. Ind on c.
$E(k,1) = k$.
We show $E(k,c) \le W(kX,c)$ for a **large** X.
$\mathrm{COL}\colon [W(kX,c)] \to [c]$.
By VDW there exists A, D:
$A, A+D, \ldots, A+kXD$ is color (we can assume) c.
$A, A+D, \ldots, A+(k-1)D$ are color c . So $\mathrm{COL}(D) \ne c$.

Proof of Extended VDW Thm

Pf. Ind on c.
$E(k,1) = k$.
We show $E(k,c) \le W(kX,c)$ for a **large** X.
$\mathrm{COL}\colon [W(kX,c)] \to [c]$.
By VDW there exists A, D:
$A, A+D, \ldots, A+kXD$ is color (we can assume) c.
$A, A+D, \ldots, A+(k-1)D$ are color c . So $\mathrm{COL}(D) \ne c$.
$A, A+2D, \ldots, A+2(k-1)D$ are c . So $\mathrm{COL}(2D) \ne c$.

Proof of Extended VDW Thm

Pf. Ind on c.
$E(k,1)=k$.
We show $E(k,c) \le W(kX,c)$ for a **large** X.
$\mathrm{COL}\colon [W(kX,c)] \rightarrow [c]$.
By VDW there exists A, D:
$A, A+D, \ldots, A+kXD$ is color (we can assume) c.
$A, A+D, \ldots, A+(k-1)D$ are color c . So $\mathrm{COL}(D) \ne c$.
$A, A+2D, \ldots, A+2(k-1)D$ are c . So $\mathrm{COL}(2D) \ne c$.

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$A, A+XD, A+2XD, \ldots, A+(k-1)XD$. So
$\mathrm{COL}((k-1)XD) \ne c$.

Proof of Extended VDW Thm

Pf. Ind on c.
$E(k,1) = k$.
We show $E(k,c) \le W(kX,c)$ for a **large** X.
$\mathrm{COL}\colon [W(kX,c)] \rightarrow [c]$.
By VDW there exists A, D:
$A, A+D, \ldots, A+kXD$ is color (we can assume) c.
$A, A+D, \ldots, A+(k-1)D$ are color c . So $\mathrm{COL}(D) \ne c$.
$A, A+2D, \ldots, A+2(k-1)D$ are c . So $\mathrm{COL}(2D) \ne c$.

$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$

$A, A+XD, A+2XD, \ldots, A+(k-1)XD$. So
$\mathrm{COL}((k-1)XD) \ne c$.
$D, 2D, \ldots, (k-1)XD$ use $[c-1]$, only $c-1$ colors.

Proof of Extended VDW Thm (cont)

$D, 2D, \ldots, (k-1)XD$ use $[c-1]$.

Proof of Extended VDW Thm (cont)

$D, 2D, \ldots, (k-1)XD$ use $[c-1]$.

Set $X = E(k, c-1)$. This is where we use Ind. Hyp.

Proof of Extended VDW Thm (cont)

$D, 2D, \ldots, (k-1)XD$ use $[c-1]$.

Set $X = E(k, c-1)$. This is where we use Ind. Hyp.

$D, 2D, \ldots, E(k, c-1)D$ only use $[c-1]$.

Proof of Extended VDW Thm (cont)

$D, 2D, \ldots, (k-1)XD$ use $[c-1]$.

Set $X = E(k, c-1)$. This is where we use Ind. Hyp.

$D, 2D, \ldots, E(k, c-1)D$ only use $[c-1]$.
Define $\mathrm{COL}'(i) = \mathrm{COL}(iD)$, a $(c-1)$-coloring, so there exists a', d'

Proof of Extended VDW Thm (cont)

$D, 2D, \ldots, (k-1)XD$ use $[c-1]$.

Set $X = E(k, c-1)$. This is where we use Ind. Hyp.

$D, 2D, \ldots, E(k, c-1)D$ only use $[c-1]$.
Define $\mathrm{COL}'(i) = \mathrm{COL}(iD)$, a $(c-1)$-coloring, so there exists a', d'

$a', a'+d', \ldots, a'+(k-1)d', d'$ same COL' color.

Proof of Extended VDW Thm (cont)

$D, 2D, \ldots, (k-1)XD$ use $[c-1]$.

Set $X = E(k, c-1)$. This is where we use Ind. Hyp.

$D, 2D, \ldots, E(k, c-1)D$ only use $[c-1]$.
Define $\mathrm{COL}'(i) = \mathrm{COL}(iD)$, a $(c-1)$-coloring, so there exists a', d'

$a', a'+d', \ldots, a'+(k-1)d', d'$ same COL' color.

$a'D, (a'+d')D, \ldots, (a'+(k-1)d')D, d'D$ same COL color.

Proof of Extended VDW Thm (cont)

$D, 2D, \ldots, (k-1)XD$ use $[c-1]$.

Set $X = E(k, c-1)$. This is where we use Ind. Hyp.

$D, 2D, \ldots, E(k, c-1)D$ only use $[c-1]$.
Define $\mathrm{COL}'(i) = \mathrm{COL}(iD)$, a $(c-1)$-coloring, so there exists a', d'

$a', a'+d', \ldots, a'+(k-1)d', d'$ same COL' color.

$a'D, (a'+d')D, \ldots, (a'+(k-1)d')D, d'D$ same COL color.

$a'D, a'D+d'D, \ldots, a'D+(k-1)d'D, d'D$ same COL color.

Proof of Extended VDW Thm (cont)

$D, 2D, \ldots, (k-1)XD$ use $[c-1]$.

Set $X = E(k, c-1)$. This is where we use Ind. Hyp.

$D, 2D, \ldots, E(k, c-1)D$ only use $[c-1]$.
Define $\mathrm{COL}'(i) = \mathrm{COL}(iD)$, a $(c-1)$-coloring, so there exists a', d'

$a', a'+d', \ldots, a'+(k-1)d', d'$ same COL' color.

$a'D, (a'+d')D, \ldots, (a'+(k-1)d')D, d'D$ same COL color.

$a'D, a'D+d'D, \ldots, a'D+(k-1)d'D, d'D$ same COL color.
$a = a'D,\ d = d'D$

Proof of Extended VDW Thm (cont)

$D, 2D, \ldots, (k-1)XD$ use $[c-1]$.

Set $X = E(k, c-1)$. This is where we use Ind. Hyp.

$D, 2D, \ldots, E(k, c-1)D$ only use $[c-1]$.
Define $\mathrm{COL}'(i) = \mathrm{COL}(iD)$, a $(c-1)$-coloring, so there exists a', d'

$a', a'+d', \ldots, a'+(k-1)d', d'$ same COL' color.

$a'D, (a'+d')D, \ldots, (a'+(k-1)d')D, d'D$ same COL color.

$a'D, a'D+d'D, \ldots, a'D+(k-1)d'D, d'D$ same COL color.
$a = a'D,\ d = d'D$

$a, a+d, \ldots, a+(k-1)d, d$ same COL color.

Real EVDW

What I presented above is NOT the EVDW. This is:

Real EVDW

What I presented above is NOT the EVDW. This is:

EVDW Thm

$(\forall k, c, e \in \mathbb{N})(\exists E = E(k, e, c))(\forall \mathrm{COL}\colon [E] \to [c])(\exists a, d)$ st

Real EVDW

What I presented above is NOT the EVDW. This is:

EVDW Thm

$(\forall k, c, e \in \mathbb{N})(\exists E = E(k, e, c)(\forall \text{COL}\colon [E] \rightarrow [c])(\exists a, d)$ st

$$a, a+d, a+2d, \ldots, a+(k-1)d, de$$

are all the same color.

Real EVDW

What I presented above is NOT the EVDW. This is:

EVDW Thm

$(\forall k, c, e \in \mathbb{N})(\exists E = E(k, e, c)(\forall \mathrm{COL}\colon [E] \rightarrow [c])(\exists a, d)$ st

$$a, a+d, a+2d, \ldots, a+(k-1)d, de$$

are all the same color.

This I leave to the reader.

Real EVDW

What I presented above is NOT the EVDW. This is:

EVDW Thm

$(\forall k, c, e \in \mathbb{N})(\exists E = E(k, e, c)(\forall \mathrm{COL}\colon [E] \rightarrow [c])(\exists a, d)$ st

$$a, a + d, a + 2d, \ldots, a + (k - 1)d, de$$

are all the same color.
This I leave to the reader.
We will only use the $e = 1$ case.

Back to

$$2w + 3x = 5y + z$$

$2w + 3x = 5y + z$

Thm $2w + 3x = 5y + z$ is regular.

$2w + 3x = 5y + z$

Thm $2w + 3x = 5y + z$ is regular.
Let $c \in \mathbb{N}$.Use EVDW's thm with c and with k we pick later.

$2w + 3x = 5y + z$

Thm $2w + 3x = 5y + z$ is regular.
Let $c \in \mathbb{N}$.Use EVDW's thm with c and with k we pick later.
$\exists W$ for all $\mathrm{COL}[W] \rightarrow [c]$ $\exists a, d$

$2w + 3x = 5y + z$

Thm $2w + 3x = 5y + z$ is regular.
Let $c \in \mathbb{N}$.Use EVDW's thm with c and with k we pick later.
$\exists W$ for all $\mathrm{COL}[W] \rightarrow [c]$ $\exists a, d$

$$a, a+d, \ldots, a+(k-1)d, d \text{ are all the same color}$$

$2w + 3x = 5y + z$

Thm $2w + 3x = 5y + z$ is regular.
Let $c \in \mathbb{N}$.Use EVDW's thm with c and with k we pick later.
$\exists W$ for all $\mathrm{COL}[W] \rightarrow [c]$ $\exists a, d$

$$a, a+d, \ldots, a+(k-1)d, d \text{ are all the same color}$$

We pick $0 \le W, X, Y, Z \le k$ later and then set

$2w + 3x = 5y + z$

Thm $2w + 3x = 5y + z$ is regular.
Let $c \in \mathbb{N}$.Use EVDW's thm with c and with k we pick later.
$\exists W$ for all $\mathrm{COL}[W] \rightarrow [c]$ $\exists a, d$

$$a, a + d, \ldots, a + (k-1)d, d \text{ are all the same color}$$

We pick $0 \le W, X, Y, Z \le k$ later and then set
$w = a + Wd$

$2w + 3x = 5y + z$

Thm $2w + 3x = 5y + z$ is regular.
Let $c \in \mathbb{N}$.Use EVDW's thm with c and with k we pick later.
$\exists W$ for all $\mathrm{COL}[W] \rightarrow [c]$ $\exists a, d$

$$a, a+d, \ldots, a+(k-1)d, d \text{ are all the same color}$$

We pick $0 \le W, X, Y, Z \le k$ later and then set
$w = a + Wd \qquad x = a + Xd$

$2w + 3x = 5y + z$

Thm $2w + 3x = 5y + z$ is regular.
Let $c \in \mathbb{N}$.Use EVDW's thm with c and with k we pick later.
$\exists W$ for all $\mathrm{COL}[W] \rightarrow [c]$ $\exists a, d$

$$a, a+d, \ldots, a+(k-1)d, d \text{ are all the same color}$$

We pick $0 \le W, X, Y, Z \le k$ later and then set
$w = a + Wd \qquad x = a + Xd \qquad y = a + Yd$

$2w + 3x = 5y + z$

Thm $2w + 3x = 5y + z$ is regular.
Let $c \in \mathbb{N}$.Use EVDW's thm with c and with k we pick later.
$\exists W$ for all $\mathrm{COL}[W] \to [c]$ $\exists a, d$

$$a, a + d, \ldots, a + (k-1)d, d \text{ are all the same color}$$

We pick $0 \le W, X, Y, Z \le k$ later and then set
$w = a + Wd \qquad x = a + Xd \qquad y = a + Yd \qquad z = Zd$

$2w + 3x = 5y + z$

Thm $2w + 3x = 5y + z$ is regular.
Let $c \in \mathbb{N}$.Use EVDW's thm with c and with k we pick later.
$\exists W$ for all $\mathrm{COL}[W] \rightarrow [c]$ $\exists a, d$

$$a, a + d, \ldots, a + (k-1)d, d \text{ are all the same color}$$

We pick $0 \le W, X, Y, Z \le k$ later and then set
$w = a + Wd$ $\quad x = a + Xd$ $\quad y = a + Yd$ $\quad z = Zd$
Good News: $\mathrm{COL}(w) = \mathrm{COL}(x) = \mathrm{COL}(y) = \mathrm{COL}(z)$.

$2w + 3x = 5y + z$

Thm $2w + 3x = 5y + z$ is regular.
Let $c \in \mathbb{N}$.Use EVDW's thm with c and with k we pick later.
$\exists W$ for all $\mathrm{COL}[W] \rightarrow [c]$ $\exists a, d$

$$a, a+d, \ldots, a+(k-1)d, d \text{ are all the same color}$$

We pick $0 \le W, X, Y, Z \le k$ later and then set
$w = a + Wd$ $\quad$ $x = a + Xd$ $\quad$ $y = a + Yd$ $\quad$ $z = Zd$
Good News: $\mathrm{COL}(w) = \mathrm{COL}(x) = \mathrm{COL}(y) = \mathrm{COL}(z)$.
Want

$2w + 3x = 5y + z$

Thm $2w + 3x = 5y + z$ is regular.
Let $c \in \mathbb{N}$.Use EVDW's thm with c and with k we pick later.
$\exists W$ for all $\mathrm{COL}[W] \rightarrow [c]$ $\exists a, d$

$$a, a+d, \ldots, a+(k-1)d, d \text{ are all the same color}$$

We pick $0 \le W, X, Y, Z \le k$ later and then set
$w = a + Wd$ $\quad x = a + Xd$ $\quad y = a + Yd$ $\quad z = Zd$
Good News: $\mathrm{COL}(w) = \mathrm{COL}(x) = \mathrm{COL}(y) = \mathrm{COL}(z)$.
Want

$$2w + 3x = 5y + z$$

$2w + 3x = 5y + z$

Thm $2w + 3x = 5y + z$ is regular.
Let $c \in \mathbb{N}$.Use EVDW's thm with c and with k we pick later.
$\exists W$ for all $\mathrm{COL}[W] \rightarrow [c]$ $\exists a, d$

$$a, a+d, \ldots, a+(k-1)d, d \text{ are all the same color}$$

We pick $0 \le W, X, Y, Z \le k$ later and then set
$w = a + Wd$ $\quad x = a + Xd$ $\quad y = a + Yd$ $\quad z = Zd$
Good News: $\mathrm{COL}(w) = \mathrm{COL}(x) = \mathrm{COL}(y) = \mathrm{COL}(z)$.
Want

$$2w + 3x = 5y + z$$

$$2(a + Wd) + 3(a + Xd) = 5(a + Yd) + (Zd)$$

How We Get What We Want

How We Get What We Want

Want

$2w + 3x = 5y + z$

How We Get What We Want

Want

$2w + 3x = 5y + z$

$2(a + Wd) + 3(a + Xd) = 5(a + Yd) + (Zd)$

How We Get What We Want

Want

$2w + 3x = 5y + z$

$2(a + Wd) + 3(a + Xd) = 5(a + Yd) + (Zd)$

$2a + 2Wd + 3a + 3Xd = 5a + 5Yd + Zd$ **WOW** The a's drop out.

How We Get What We Want

Want

$2w + 3x = 5y + z$

$2(a + Wd) + 3(a + Xd) = 5(a + Yd) + (Zd)$

$2a + 2Wd + 3a + 3Xd = 5a + 5Yd + Zd$ **WOW** The a's drop out.

$2Wd + 3Xd = 5Yd + Zd$ **WOW** The d's drop out.

How We Get What We Want

Want

$2w + 3x = 5y + z$

$2(a + Wd) + 3(a + Xd) = 5(a + Yd) + (Zd)$

$2a + 2Wd + 3a + 3Xd = 5a + 5Yd + Zd$ **WOW** The a's drop out.

$2Wd + 3Xd = 5Yd + Zd$ **WOW** The d's drop out.

$2W + 3X = 5Y + Z$

How We Get What We Want

Want

$2w + 3x = 5y + z$

$2(a + Wd) + 3(a + Xd) = 5(a + Yd) + (Zd)$

$2a + 2Wd + 3a + 3Xd = 5a + 5Yd + Zd$ **WOW** The a's drop out.

$2Wd + 3Xd = 5Yd + Zd$ **WOW** The d's drop out.

$2W + 3X = 5Y + Z$

We'll take $W = 2,\ X = 4,\ Y = 3,\ Z = 1$

How We Get What We Want

Want

$2w + 3x = 5y + z$

$2(a + Wd) + 3(a + Xd) = 5(a + Yd) + (Zd)$

$2a + 2Wd + 3a + 3Xd = 5a + 5Yd + Zd$ **WOW** The a's drop out.

$2Wd + 3Xd = 5Yd + Zd$ **WOW** The d's drop out.

$2W + 3X = 5Y + Z$

We'll take $W = 2,\ X = 4,\ Y = 3,\ Z = 1$

So take $w = a + 2d \qquad x = a + 4d \qquad y = a + 3d \qquad z = d$

How We Get What We Want

Want

$2w + 3x = 5y + z$

$2(a + Wd) + 3(a + Xd) = 5(a + Yd) + (Zd)$

$2a + 2Wd + 3a + 3Xd = 5a + 5Yd + Zd$ **WOW** The a's drop out.

$2Wd + 3Xd = 5Yd + Zd$ **WOW** The d's drop out.

$2W + 3X = 5Y + Z$

We'll take $W = 2,\ X = 4,\ Y = 3,\ Z = 1$

So take $w = a + 2d \qquad x = a + 4d \qquad y = a + 3d \qquad z = d$

So take EVDW with $k = 5$.

How We Get What We Want

Want

$2w + 3x = 5y + z$

$2(a + Wd) + 3(a + Xd) = 5(a + Yd) + (Zd)$

$2a + 2Wd + 3a + 3Xd = 5a + 5Yd + Zd$ **WOW** The a's drop out.

$2Wd + 3Xd = 5Yd + Zd$ **WOW** The d's drop out.

$2W + 3X = 5Y + Z$

We'll take $W = 2,\ X = 4,\ Y = 3,\ Z = 1$

So take $w = a + 2d \qquad x = a + 4d \qquad y = a + 3d \qquad z = d$

So take EVDW with $k = 5$.

Done

Rado's Thm (Half of it)

Thm Let $a_1, \ldots, a_k \in \mathbb{Z}$ be st some subset of the a_i's sums to 0. Then

Rado's Thm (Half of it)

Thm Let $a_1, \ldots, a_k \in \mathbb{Z}$ be st some subset of the a_i's sums to 0. Then
$a_1 x_1 + \cdots + a_k x_k = 0$ is regular.

Rado's Thm (Half of it)

Thm Let $a_1, \ldots, a_k \in \mathbb{Z}$ be st some subset of the a_i's sums to 0. Then
$a_1x_1 + \cdots + a_kx_k = 0$ is regular.

We won't prove. You have seen most of the ideas needed to prove it.

An Equation Where Rado Fails

$x + 2y = 4z$

We define $\mathrm{COL}\colon \mathbb{N} \to [4]$ st

$x + 2y = 4z$

We define $\mathrm{COL}\colon \mathbb{N} \rightarrow [4]$ st

$$x + 2y = 4z \text{ has no mono solution.}$$

$x + 2y = 4z$

We define $\mathrm{COL}\colon \mathbb{N} \rightarrow [4]$ st

$$x + 2y = 4z \text{ has no mono solution.}$$

$\mathrm{COL}(5^a b) = b \bmod 5$. Note that $b \neq 0$.

$x + 2y = 4z$

We define $\mathrm{COL}\colon \mathbb{N} \rightarrow [4]$ st

$$x + 2y = 4z \text{ has no mono solution.}$$

$\mathrm{COL}(5^a b) = b \bmod 5$. Note that $b \neq 0$.

If a_1, a_2, a_3 is a mono solution, say color is b.

$x + 2y = 4z$

We define $\mathrm{COL}\colon \mathbb{N} \to [4]$ st

$$x + 2y = 4z \text{ has no mono solution.}$$

$\mathrm{COL}(5^a b) = b \bmod 5$. Note that $b \neq 0$.

If a_1, a_2, a_3 is a mono solution, say color is b.

$$a_1 = 5^{e_1} b_1 \qquad a_2 = 5^{e_2} b_2 \qquad a_3 = 5^{e_3} b_3$$

$x + 2y = 4z$

We define $\mathrm{COL}\colon \mathbb{N} \rightarrow [4]$ st

$$x + 2y = 4z \text{ has no mono solution.}$$

$\mathrm{COL}(5^a b) = b \bmod 5$. Note that $b \neq 0$.

If a_1, a_2, a_3 is a mono solution, say color is b.

$$a_1 = 5^{e_1} b_1 \qquad a_2 = 5^{e_2} b_2 \qquad a_3 = 5^{e_3} b_3$$

$b_1 \equiv b_2 \equiv b_3 \equiv b \pmod 5$

Case $e_1 < e_2, e_3$

$$a_1 = 5^{e_1} b_1 \qquad a_2 = 5^{e_2} b_2 \qquad a_3 = 5^{e_3} b_3$$

Case $e_1 < e_2, e_3$

$$a_1 = 5^{e_1} b_1 \qquad a_2 = 5^{e_2} b_2 \qquad a_3 = 5^{e_3} b_3$$

$b_1 \equiv b_2 \equiv b_3 \equiv b \pmod 5$ and $e_1 < e_2, e_3$.

Case $e_1 < e_2, e_3$

$$a_1 = 5^{e_1} b_1 \qquad a_2 = 5^{e_2} b_2 \qquad a_3 = 5^{e_3} b_3$$

$b_1 \equiv b_2 \equiv b_3 \equiv b \pmod 5$ and $e_1 < e_2, e_3$.
Recall $b \neq 0$.

Case $e_1 < e_2, e_3$

$$a_1 = 5^{e_1} b_1 \qquad a_2 = 5^{e_2} b_2 \qquad a_3 = 5^{e_3} b_3$$

$b_1 \equiv b_2 \equiv b_3 \equiv b \pmod 5$ and $e_1 < e_2, e_3$.
Recall $b \neq 0$.

$$a_1 + 2a_2 = 4a_3$$

Case $e_1 < e_2, e_3$

$$a_1 = 5^{e_1} b_1 \qquad a_2 = 5^{e_2} b_2 \qquad a_3 = 5^{e_3} b_3$$

$b_1 \equiv b_2 \equiv b_3 \equiv b \pmod 5$ and $e_1 < e_2, e_3$.
Recall $b \neq 0$.

$$a_1 + 2a_2 = 4a_3$$

$$5^{e_1} b_1 + 2 \times 5^{e_2} b_2 = 4 \times 5^{e_3} b_3$$

Case $e_1 < e_2, e_3$

$$a_1 = 5^{e_1} b_1 \qquad a_2 = 5^{e_2} b_2 \qquad a_3 = 5^{e_3} b_3$$

$b_1 \equiv b_2 \equiv b_3 \equiv b \pmod 5$ and $e_1 < e_2, e_3$.
Recall $b \neq 0$.

$$a_1 + 2a_2 = 4a_3$$

$$5^{e_1} b_1 + 2 \times 5^{e_2} b_2 = 4 \times 5^{e_3} b_3$$

Divide by 5^{e_1} to get:

Case $e_1 < e_2, e_3$

$$a_1 = 5^{e_1} b_1 \qquad a_2 = 5^{e_2} b_2 \qquad a_3 = 5^{e_3} b_3$$

$b_1 \equiv b_2 \equiv b_3 \equiv b \pmod 5$ and $e_1 < e_2, e_3$.
Recall $b \neq 0$.

$$a_1 + 2a_2 = 4a_3$$

$$5^{e_1} b_1 + 2 \times 5^{e_2} b_2 = 4 \times 5^{e_3} b_3$$

Divide by 5^{e_1} to get:

$$b_1 + 2 \times 5^{e_2 - e_1} b_2 = 4 \times 5^{e_3 - e_1} b_3$$

Case $e_1 < e_2, e_3$

$$a_1 = 5^{e_1} b_1 \qquad a_2 = 5^{e_2} b_2 \qquad a_3 = 5^{e_3} b_3$$

$b_1 \equiv b_2 \equiv b_3 \equiv b \pmod 5$ and $e_1 < e_2, e_3$.
Recall $b \neq 0$.

$$a_1 + 2a_2 = 4a_3$$

$$5^{e_1} b_1 + 2 \times 5^{e_2} b_2 = 4 \times 5^{e_3} b_3$$

Divide by 5^{e_1} to get:

$$b_1 + 2 \times 5^{e_2 - e_1} b_2 = 4 \times 5^{e_3 - e_1} b_3$$

Take this mod 5 to get:

Case $e_1 < e_2, e_3$

$$a_1 = 5^{e_1} b_1 \qquad a_2 = 5^{e_2} b_2 \qquad a_3 = 5^{e_3} b_3$$

$b_1 \equiv b_2 \equiv b_3 \equiv b \pmod 5$ and $e_1 < e_2, e_3$.
Recall $b \neq 0$.

$$a_1 + 2a_2 = 4a_3$$

$$5^{e_1} b_1 + 2 \times 5^{e_2} b_2 = 4 \times 5^{e_3} b_3$$

Divide by 5^{e_1} to get:

$$b_1 + 2 \times 5^{e_2 - e_1} b_2 = 4 \times 5^{e_3 - e_1} b_3$$

Take this mod 5 to get:

$$b \equiv 0 \pmod 5 \text{ contradiction}$$

Case $e_2 < e_1, e_3$

$$a_1 = 5^{e_1} b_1 \qquad a_2 = 5^{e_2} b_2 \qquad a_3 = 5^{e_3} b_3$$

Case $e_2 < e_1, e_3$

$$a_1 = 5^{e_1} b_1 \qquad a_2 = 5^{e_2} b_2 \qquad a_3 = 5^{e_3} b_3$$

$b_1 \equiv b_2 \equiv b_3 \equiv b \pmod 5$ and $e_2 < e_1, e_3$.

Case $e_2 < e_1, e_3$

$$a_1 = 5^{e_1} b_1 \qquad a_2 = 5^{e_2} b_2 \qquad a_3 = 5^{e_3} b_3$$

$b_1 \equiv b_2 \equiv b_3 \equiv b \pmod 5$ and $e_2 < e_1, e_3$. Recall $b \neq 0$.

Case $e_2 < e_1, e_3$

$$a_1 = 5^{e_1} b_1 \qquad a_2 = 5^{e_2} b_2 \qquad a_3 = 5^{e_3} b_3$$

$b_1 \equiv b_2 \equiv b_3 \equiv b \pmod 5$ and $e_2 < e_1, e_3$. Recall $b \neq 0$.

$$a_1 + 2a_2 = 4a_3$$

Case $e_2 < e_1, e_3$

$$a_1 = 5^{e_1} b_1 \qquad a_2 = 5^{e_2} b_2 \qquad a_3 = 5^{e_3} b_3$$

$b_1 \equiv b_2 \equiv b_3 \equiv b \pmod 5$ and $e_2 < e_1, e_3$. Recall $b \neq 0$.

$$a_1 + 2a_2 = 4a_3$$

$$5^{e_1} b_1 + 2 \times 5^{e_2} b_2 = 4 \times 5^{e_3} b_3$$

Case $e_2 < e_1, e_3$

$$a_1 = 5^{e_1} b_1 \qquad a_2 = 5^{e_2} b_2 \qquad a_3 = 5^{e_3} b_3$$

$b_1 \equiv b_2 \equiv b_3 \equiv b \pmod 5$ and $e_2 < e_1, e_3$. Recall $b \neq 0$.

$$a_1 + 2a_2 = 4a_3$$

$$5^{e_1} b_1 + 2 \times 5^{e_2} b_2 = 4 \times 5^{e_3} b_3$$

Divide by 5^{e_2} to get:

Case $e_2 < e_1, e_3$

$$a_1 = 5^{e_1} b_1 \qquad a_2 = 5^{e_2} b_2 \qquad a_3 = 5^{e_3} b_3$$

$b_1 \equiv b_2 \equiv b_3 \equiv b \pmod 5$ and $e_2 < e_1, e_3$. Recall $b \neq 0$.

$$a_1 + 2a_2 = 4a_3$$

$$5^{e_1} b_1 + 2 \times 5^{e_2} b_2 = 4 \times 5^{e_3} b_3$$

Divide by 5^{e_2} to get:

$$5^{e_1 - e_2} b_1 + 2 \times b_2 = 4 \times 5^{e_3 - e_2} b_3$$

Case $e_2 < e_1, e_3$

$$a_1 = 5^{e_1} b_1 \qquad a_2 = 5^{e_2} b_2 \qquad a_3 = 5^{e_3} b_3$$

$b_1 \equiv b_2 \equiv b_3 \equiv b \pmod 5$ and $e_2 < e_1, e_3$. Recall $b \neq 0$.

$$a_1 + 2a_2 = 4a_3$$

$$5^{e_1} b_1 + 2 \times 5^{e_2} b_2 = 4 \times 5^{e_3} b_3$$

Divide by 5^{e_2} to get:

$$5^{e_1 - e_2} b_1 + 2 \times b_2 = 4 \times 5^{e_3 - e_2} b_3$$

Take this mod 5 to get:

Case $e_2 < e_1, e_3$

$$a_1 = 5^{e_1} b_1 \qquad a_2 = 5^{e_2} b_2 \qquad a_3 = 5^{e_3} b_3$$

$b_1 \equiv b_2 \equiv b_3 \equiv b \pmod 5$ and $e_2 < e_1, e_3$. Recall $b \neq 0$.

$$a_1 + 2a_2 = 4a_3$$

$$5^{e_1} b_1 + 2 \times 5^{e_2} b_2 = 4 \times 5^{e_3} b_3$$

Divide by 5^{e_2} to get:

$$5^{e_1 - e_2} b_1 + 2 \times b_2 = 4 \times 5^{e_3 - e_2} b_3$$

Take this mod 5 to get:

$$2b \equiv 0 \pmod 5 \text{ contradiction}$$

Case $e_2 < e_1, e_3$

$$a_1 = 5^{e_1} b_1 \qquad a_2 = 5^{e_2} b_2 \qquad a_3 = 5^{e_3} b_3$$

$b_1 \equiv b_2 \equiv b_3 \equiv b \pmod 5$ and $e_2 < e_1, e_3$. Recall $b \neq 0$.

$$a_1 + 2a_2 = 4a_3$$

$$5^{e_1} b_1 + 2 \times 5^{e_2} b_2 = 4 \times 5^{e_3} b_3$$

Divide by 5^{e_2} to get:

$$5^{e_1 - e_2} b_1 + 2 \times b_2 = 4 \times 5^{e_3 - e_2} b_3$$

Take this mod 5 to get:

$$2b \equiv 0 \pmod 5 \text{ contradiction}$$

Key 5 is prime: $2b \equiv 0 \pmod 5$ implies $b \equiv 0 \pmod 5$.

Case $e_2 < e_1, e_3$

$$a_1 = 5^{e_1} b_1 \qquad a_2 = 5^{e_2} b_2 \qquad a_3 = 5^{e_3} b_3$$

$b_1 \equiv b_2 \equiv b_3 \equiv b \pmod 5$ and $e_2 < e_1, e_3$. Recall $b \not\equiv 0$.

$$a_1 + 2a_2 = 4a_3$$

$$5^{e_1} b_1 + 2 \times 5^{e_2} b_2 = 4 \times 5^{e_3} b_3$$

Divide by 5^{e_2} to get:

$$5^{e_1 - e_2} b_1 + 2 \times b_2 = 4 \times 5^{e_3 - e_2} b_3$$

Take this mod 5 to get:

$$2b \equiv 0 \pmod 5 \text{ contradiction}$$

Key 5 is prime: $2b \equiv 0 \pmod 5$ implies $b \equiv 0 \pmod 5$. Contradiction

Case $e_3 < e_1, e_2$

Similar to $e_2 < e_1, 3_3$.

Case $e_1 = e_2 < e_3$

$$a_1 = 5^{e_1} b_1 \qquad a_2 = 5^{e_2} b_2 \qquad a_3 = 5^{e_3} b_3$$

Case $e_1 = e_2 < e_3$

$$a_1 = 5^{e_1} b_1 \qquad a_2 = 5^{e_2} b_2 \qquad a_3 = 5^{e_3} b_3$$

$b_1 \equiv b_2 \equiv b_3 \equiv b \pmod 5$ and $e_2 < e_1, e_3$.

Case $e_1 = e_2 < e_3$

$$a_1 = 5^{e_1} b_1 \qquad a_2 = 5^{e_2} b_2 \qquad a_3 = 5^{e_3} b_3$$

$b_1 \equiv b_2 \equiv b_3 \equiv b \pmod 5$ and $e_2 < e_1, e_3$. Recall $b \neq 0$.

$$a_1 + 2a_2 = 4a_3$$

Case $e_1 = e_2 < e_3$

$$a_1 = 5^{e_1} b_1 \qquad a_2 = 5^{e_2} b_2 \qquad a_3 = 5^{e_3} b_3$$

$b_1 \equiv b_2 \equiv b_3 \equiv b \pmod 5$ and $e_2 < e_1, e_3$. Recall $b \neq 0$.

$$a_1 + 2a_2 = 4a_3$$

$$5^{e_1} b_1 + 2 \times 5^{e_1} b_2 = 4 \times 5^{e_3} b_3$$

Case $e_1 = e_2 < e_3$

$$a_1 = 5^{e_1} b_1 \qquad a_2 = 5^{e_2} b_2 \qquad a_3 = 5^{e_3} b_3$$

$b_1 \equiv b_2 \equiv b_3 \equiv b \pmod 5$ and $e_2 < e_1, e_3$. Recall $b \neq 0$.

$$a_1 + 2a_2 = 4a_3$$

$$5^{e_1} b_1 + 2 \times 5^{e_1} b_2 = 4 \times 5^{e_3} b_3$$

Divide by 5^{e_1} to get:

$$b_1 + 2b_2 = 4 \times 5^{e_3 - e_1} b_3$$

Case $e_1 = e_2 < e_3$

$$a_1 = 5^{e_1} b_1 \qquad a_2 = 5^{e_2} b_2 \qquad a_3 = 5^{e_3} b_3$$

$b_1 \equiv b_2 \equiv b_3 \equiv b \pmod 5$ and $e_2 < e_1, e_3$. Recall $b \neq 0$.

$$a_1 + 2a_2 = 4a_3$$

$$5^{e_1} b_1 + 2 \times 5^{e_1} b_2 = 4 \times 5^{e_3} b_3$$

Divide by 5^{e_1} to get:

$$b_1 + 2b_2 = 4 \times 5^{e_3 - e_1} b_3$$

Take this mod 5 to get:

$$b + 2b \equiv 0 \pmod 5$$

Case $e_1 = e_2 < e_3$

$$a_1 = 5^{e_1} b_1 \qquad a_2 = 5^{e_2} b_2 \qquad a_3 = 5^{e_3} b_3$$

$b_1 \equiv b_2 \equiv b_3 \equiv b \pmod 5$ and $e_2 < e_1, e_3$. Recall $b \neq 0$.

$$a_1 + 2a_2 = 4a_3$$

$$5^{e_1} b_1 + 2 \times 5^{e_1} b_2 = 4 \times 5^{e_3} b_3$$

Divide by 5^{e_1} to get:

$$b_1 + 2b_2 = 4 \times 5^{e_3 - e_1} b_3$$

Take this mod 5 to get:

$$b + 2b \equiv 0 \pmod 5$$

$3b \equiv 0 \pmod 5$

Case $e_1 = e_2 < e_3$

$$a_1 = 5^{e_1} b_1 \qquad a_2 = 5^{e_2} b_2 \qquad a_3 = 5^{e_3} b_3$$

$b_1 \equiv b_2 \equiv b_3 \equiv b \pmod 5$ and $e_2 < e_1, e_3$. Recall $b \neq 0$.

$$a_1 + 2a_2 = 4a_3$$

$$5^{e_1} b_1 + 2 \times 5^{e_1} b_2 = 4 \times 5^{e_3} b_3$$

Divide by 5^{e_1} to get:

$$b_1 + 2b_2 = 4 \times 5^{e_3 - e_1} b_3$$

Take this mod 5 to get:

$$b + 2b \equiv 0 \pmod 5$$

$3b \equiv 0 \pmod 5$
$3b \equiv 0 \pmod 5$ implies $b \equiv 0 \pmod 5$. Contradiction.

Case $e_1 = e_3 < e_2$ and $e_2 = e_3 < e_1$

Similar to $e_1 = e_2 < e_3$.

Case $e_1 = e_2 = e_3$

$$a_1 = 5^{e_1} b_1 \qquad a_2 = 5^{e_2} b_2 \qquad a_3 = 5^{e_3} b_3$$

Case $e_1 = e_2 = e_3$

$$a_1 = 5^{e_1} b_1 \qquad a_2 = 5^{e_2} b_2 \qquad a_3 = 5^{e_3} b_3$$

$b_1 \equiv b_2 \equiv b_3 \equiv b \pmod 5$ and $e_2 < e_1, e_3$.

Case $e_1 = e_2 = e_3$

$$a_1 = 5^{e_1} b_1 \qquad a_2 = 5^{e_2} b_2 \qquad a_3 = 5^{e_3} b_3$$

$b_1 \equiv b_2 \equiv b_3 \equiv b \pmod 5$ and $e_2 < e_1, e_3$. Recall $b \neq 0$.

$$a_1 + 2a_2 = 4a_3$$

Case $e_1 = e_2 = e_3$

$$a_1 = 5^{e_1} b_1 \qquad a_2 = 5^{e_2} b_2 \qquad a_3 = 5^{e_3} b_3$$

$b_1 \equiv b_2 \equiv b_3 \equiv b \pmod 5$ and $e_2 < e_1, e_3$. Recall $b \neq 0$.

$$a_1 + 2a_2 = 4a_3$$

$$5^{e_1} b_1 + 2 \times 5^{e_1} b_2 = 4 \times 5^{e_1} b_3$$

Case $e_1 = e_2 = e_3$

$$a_1 = 5^{e_1} b_1 \qquad a_2 = 5^{e_2} b_2 \qquad a_3 = 5^{e_3} b_3$$

$b_1 \equiv b_2 \equiv b_3 \equiv b \pmod 5$ and $e_2 < e_1, e_3$. Recall $b \neq 0$.

$$a_1 + 2a_2 = 4a_3$$

$$5^{e_1} b_1 + 2 \times 5^{e_1} b_2 = 4 \times 5^{e_1} b_3$$

Divide by 5^{e_1} to get:

Case $e_1 = e_2 = e_3$

$$a_1 = 5^{e_1} b_1 \qquad a_2 = 5^{e_2} b_2 \qquad a_3 = 5^{e_3} b_3$$

$b_1 \equiv b_2 \equiv b_3 \equiv b \pmod 5$ and $e_2 < e_1, e_3$. Recall $b \neq 0$.

$$a_1 + 2a_2 = 4a_3$$

$$5^{e_1} b_1 + 2 \times 5^{e_1} b_2 = 4 \times 5^{e_1} b_3$$

Divide by 5^{e_1} to get:

$$b_1 + 2b_2 = 4b_3$$

Case $e_1 = e_2 = e_3$

$$a_1 = 5^{e_1} b_1 \qquad a_2 = 5^{e_2} b_2 \qquad a_3 = 5^{e_3} b_3$$

$b_1 \equiv b_2 \equiv b_3 \equiv b \pmod 5$ and $e_2 < e_1, e_3$. Recall $b \neq 0$.

$$a_1 + 2a_2 = 4a_3$$

$$5^{e_1} b_1 + 2 \times 5^{e_1} b_2 = 4 \times 5^{e_1} b_3$$

Divide by 5^{e_1} to get:

$$b_1 + 2b_2 = 4b_3$$

Take this mod 5 to get $3b \equiv 4b$ so $b \equiv 0 \pmod 5$ Contradiction.

What Was Special About $x + 2y = 4z$? About 5?

What Was Special About $x + 2y = 4z$? About 5?

1. The proof used that NO subset of $1, 2, -4$ sums to 0.

What Was Special About $x + 2y = 4z$? About 5?

1. The proof used that NO subset of $1, 2, -4$ sums to 0.
2. We used 5 since

What Was Special About $x + 2y = 4z$? About 5?

1. The proof used that NO subset of $1, 2, -4$ sums to 0.
2. We used 5 since
   2.1 We need a prime p

What Was Special About $x + 2y = 4z$? About 5?

1. The proof used that NO subset of $1, 2, -4$ sums to 0.
2. We used 5 since
   2.1 We need a prime p
   2.2 We needed $3b \equiv 0 \pmod{p}$ implies $b \equiv 0 \pmod{p}$

What Was Special About $x + 2y = 4z$? About 5?

1. The proof used that NO subset of $1, 2, -4$ sums to 0.
2. We used 5 since
   2.1 We need a prime p
   2.2 We needed $3b \equiv 0 \pmod{p}$ implies $b \equiv 0 \pmod{p}$
   5 is the lowest such prime.

Rado's Thm (Other Half of it)

Thm Let $a_1, \ldots, a_k \in \mathbb{Z}$ be st no subset of the a_i's sums to 0.
$a_1x_1 + \cdots + a_kx_k = 0$
is not regular.

Rado's Thm (Other Half of it)

Thm Let $a_1, \ldots, a_k \in \mathbb{Z}$ be st no subset of the a_i's sums to 0.
$a_1x_1 + \cdots + a_kx_k = 0$
is not regular.

We will not prove this but you have all of the ideas you need to prove it.
(The c-coloring that shows non-regularity uses c=the first prime bigger then any sum of the coefficients.)

Rado's Thm (Other Half of it)

Thm Let $a_1, \ldots, a_k \in \mathbb{Z}$ be st no subset of the a_i's sums to 0.
$a_1x_1 + \cdots + a_kx_k = 0$
is not regular.

We will not prove this but you have all of the ideas you need to prove it.
(The c-coloring that shows non-regularity uses c=the first prime bigger then any sum of the coefficients.)

Research Question

1. For $x + 2y = 4z$ what about 4-coloring? 3-coloring? 2-coloring?
2. More generally one can take an equation where no sum of the coefficients is 0 and look at colorings with a small number of colors.

Full Rado

Full Rado Thm A linear equation $\sum_{i=1}^{n} a_i x_i = 0$ is regular iff some subset of the coefficient sum to 0.

Full Rado

Full Rado Thm A linear equation $\sum_{i=1}^{n} a_i x_i = 0$ is regular iff some subset of the coefficient sum to 0.

(For most equations with the coefficients sum to 0 you actually get d-regular.)

Misc

Research Questions

(Some is known about some of these.)
Prove or disprove that the equations below are regular.

Research Questions

(Some is known about some of these.)
Prove or disprove that the equations below are regular.

1. $\sum_{i=1}^{n} a_i x_i = A$ for some A.

Research Questions

(Some is known about some of these.)

Prove or disprove that the equations below are regular.

1. $\sum_{i=1}^{n} a_i x_i = A$ for some A.
2. Higher degree equations (seems hard).

Other Equations

Other Equations

1. There is a matrix form of Rado which we omit.

Other Equations

1. There is a matrix form of Rado which we omit.
2. **Folkman's Thm** For all k, c there exists $N = N(k, c)$ st for all $\mathrm{COL}\colon [N] \to [c]$ there exists $a_1, \ldots, a_k$ st ALL non-empty sums of the a_i's are the same color.

Other Equations

1. There is a matrix form of Rado which we omit.
2. **Folkman's Thm** For all k, c there exists $N = N(k, c)$ st for all $\mathrm{COL}\colon [N] \rightarrow [c]$ there exists $a_1, \ldots, a_k$ st ALL non-empty sums of the a_i's are the same color.
3. For all c there exists $N = N(c)$ st for any $\mathrm{COL}\colon [N] \rightarrow [c]$ there is a mono solution to $16x^2 + 9y^2 = z^2$.

Other Equations

1. There is a matrix form of Rado which we omit.
2. **Folkman's Thm** For all k, c there exists $N = N(k, c)$ st for all $\mathrm{COL}\colon [N]\to[c]$ there exists $a_1, \ldots, a_k$ st ALL non-empty sums of the a_i's are the same color.
3. For all c there exists $N = N(c)$ st for any $\mathrm{COL}\colon [N]\to[c]$ there is a mono solution to $16x^2 + 9y^2 = z^2$.

   (This equation has certain properties that make it work, so there is really a more general thm here.) `http://fourier.math.uoc.gr/~ergodic/Slides/Host.pdf`

$x^2 + y^2 = z^2$ Result by Heule&Kullmann&Marek

Thm There exists N st for any $\mathrm{COL}\colon [N] \rightarrow [2]$ there is a mono solution to $x^2 + y^2 = z^2$.

$x^2 + y^2 = z^2$ Result by Heule&Kullmann&Marek

Thm There exists N st for any $\mathrm{COL}\colon [N] \rightarrow [2]$ there is a mono solution to $x^2 + y^2 = z^2$.

Do we know what N is? We actually do!

$x^2 + y^2 = z^2$ Result by Heule&Kullmann&Marek

Thm There exists N st for any $\mathrm{COL}\colon [N] \rightarrow [2]$ there is a mono solution to $x^2 + y^2 = z^2$.

Do we know what N is? We actually do!

- $\exists$ 2-col of [7824] w/o mono sol to $x^2 + y^2 = z^2$.

$x^2 + y^2 = z^2$ Result by Heule&Kullmann&Marek

Thm There exists N st for any $\mathrm{COL}\colon [N] \rightarrow [2]$ there is a mono solution to $x^2 + y^2 = z^2$.

Do we know what N is? We actually do!

- $\exists$ 2-col of [7824] w/o mono sol to $x^2 + y^2 = z^2$.
- $\forall$ 2-col of [7825] $\exists$ mono sol to $x^2 + y^2 = z^2$.

$x^2 + y^2 = z^2$ Result by Heule&Kullmann&Marek

Thm There exists N st for any $\mathrm{COL}\colon [N] \rightarrow [2]$ there is a mono solution to $x^2 + y^2 = z^2$.

Do we know what N is? We actually do!

- $\exists$ 2-col of [7824] w/o mono sol to $x^2 + y^2 = z^2$.
- $\forall$ 2-col of [7825] $\exists$ mono sol to $x^2 + y^2 = z^2$.

Thm proven by SAT-Solver. 200 terabytes: longest proof ever.

$x^2 + y^2 = z^2$ Result by Heule&Kullmann&Marek

Thm There exists N st for any $\mathrm{COL}\colon [N]\to[2]$ there is a mono solution to $x^2 + y^2 = z^2$.

Do we know what N is? We actually do!

- $\exists$ 2-col of [7824] w/o mono sol to $x^2 + y^2 = z^2$.
- $\forall$ 2-col of [7825] $\exists$ mono sol to $x^2 + y^2 = z^2$.

Thm proven by SAT-Solver. 200 terabytes: longest proof ever.

Research Questions

$x^2 + y^2 = z^2$ Result by Heule&Kullmann&Marek

Thm There exists N st for any $\mathrm{COL}\colon [N]\rightarrow[2]$ there is a mono solution to $x^2 + y^2 = z^2$.

Do we know what N is? We actually do!

- $\exists$ 2-col of [7824] w/o mono sol to $x^2 + y^2 = z^2$.
- $\forall$ 2-col of [7825] $\exists$ mono sol to $x^2 + y^2 = z^2$.

Thm proven by SAT-Solver. 200 terabytes: longest proof ever.

Research Questions
1) See how large and N you can color just with your laptop. Greedy, Randomized Greedy, are worth trying. Does Rand-Greedy do better? (I think so.)

$x^2 + y^2 = z^2$ Result by Heule&Kullmann&Marek

Thm There exists N st for any $\mathrm{COL}\colon [N] \to [2]$ there is a mono solution to $x^2 + y^2 = z^2$.

Do we know what N is? We actually do!

- $\exists$ 2-col of [7824] w/o mono sol to $x^2 + y^2 = z^2$.
- $\forall$ 2-col of [7825] $\exists$ mono sol to $x^2 + y^2 = z^2$.

Thm proven by SAT-Solver. 200 terabytes: longest proof ever.

Research Questions

1) See how large and N you can color just with your laptop. Greedy, Randomized Greedy, are worth trying. Does Rand-Greedy do better? (I think so.)

2) Once you have done (1) try it out on other equations.

$x^2 + y^2 = z^2$ Result by Heule&Kullmann&Marek

Thm There exists N st for any $\mathrm{COL}\colon [N] \to [2]$ there is a mono solution to $x^2 + y^2 = z^2$.

Do we know what N is? We actually do!

- $\exists$ 2-col of [7824] w/o mono sol to $x^2 + y^2 = z^2$.
- $\forall$ 2-col of [7825] $\exists$ mono sol to $x^2 + y^2 = z^2$.

Thm proven by SAT-Solver. 200 terabytes: longest proof ever.

Research Questions

1) See how large and N you can color just with your laptop. Greedy, Randomized Greedy, are worth trying. Does Rand-Greedy do better? (I think so.)

2) Once you have done (1) try it out on other equations.

3) (Might be Hard) Obtain a human-readable proof with perhaps a much bigger N, but which can be generalized to $c = 3$ and beyond.