BILL, RECORD LECTURE!!!!

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Poly Van Der Warden's (PVDW) Theorem

Exposition by William Gasarch

January 23, 2025

Convention

Whenever I write a, d or a, d_1 or anything of that sort we are assuming $a, d \in \mathbb{N}$ and $a, d \geq 1$.

Recall VDW's Theorem

VDW's Theorem For all k, c there exists $W = W(k, c)$ such that for all $\mathrm{COL}\colon [W] \rightarrow [c]$ there exists a, d such that

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True? or is Bill lying to us? Try to find counterexamples.

Counterexample and Reformulation

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The coloring $RBRBRB \cdots$ has no two naturals 1-apart that have same color.

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Poly VDW Theorem For all $p_1, \ldots, p_k \in \mathbb{Z}[x]$ **with** $(\forall i)[p_i(0) = 0]$, and $c \in \mathbb{N}$, there exists $W = W(p_1, \ldots, p_k; c)$ such that for all $\mathrm{COL}\colon [W] \rightarrow [c]$ there exists a a, d such that

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Credit Where Credit is Due

Poly VDW theorem first proven by Bergelson and Leibman in **Polynomial Extensions of van der Waerden's and Szemeredi's Theorem** Journal of the AMS, Vol 9, 1996. Their paper is here:
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The first Elementary proof was by Walters in **Combinatorial proofs of the Polynomial Van Der Waerden Theorem and the Polynomial Hales-Jewitt Theorem** Journal of the London Math Soc., Vol 61, 2000.
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We present his proof.

Notation

$\mathbf{PVDW}(p_1(x), \ldots, p_k(x); c)$ means
There exists $W = W(p_1, \ldots, p_k; c)$ such that for all
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$\mathbf{PVDW}(p_1(x), \ldots, p_k(x))$ means
For all c there exists $W = W(p_1, \ldots, p_k; c)$ st for all
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Easy Cases

$\mathrm{PVDW}(x, 2x, 3x, \ldots, (k-1)x))$. This is VDW's Thm.

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First hard case: $\mathrm{PVDW}(x^2; 5)$.

Poly Van Der Warden's (PVDW) Theorem: $\mathrm{PVDW}(x^2)$

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$W(x^2; 5)$: The low value of 5 does not help us.
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Thm For all $c \in \mathbb{N}$ there exists $W = W(x^2;c)$ such that for all $\mathrm{COL}\colon [W] \rightarrow [c]$, there exists a, d such that

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Note None of the results or techniques for $W(ax^2 + bx; c)$ for $c \leq 4$ will help at all. Oh well.

We Prove a Lemma Which Implies Theorem

Want:
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Lemma Fix $c \in \mathbb{N}$. For all r there exists $U = U(r)$ st for all
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We need to prove $U(r + 1)$ exists.
GOTO WHITE BOARD to prove

$$U(r + 1) \leq$$

$$(U(r)W(2U(r), c^{U(r)}))^2 + U(r)W(2U(r), c^{U(r)}).$$

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Keep that in mind.

Poly Van Der Warden's (PVDW) Theorem: $\mathrm{PVDW}(x^2 + x)$

Exposition by William Gasarch

January 23, 2025

We Begin Proof of $\mathrm{PVDW}(x^2+x)$

Thm For all $c \in \mathbb{N}$ there exists $W = W(x^2+x;c)$ st, for all $\mathrm{COL}\colon [W] \rightarrow [c]$, there exists a, d st

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This Generalizes

Thm Let $A, B \in \mathbb{Z}$. For all $c \in \mathbb{N}$ there exists $W = W(Ax^2 + Bx; c)$ st for all $\mathrm{COL}\colon [W] \rightarrow [c]$, $(\exists a, d)[a, a + Ad^2 + Bd$ same color$]$.

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Hence our bds are a byte lower than bds in real proof.

The Issue and Our Convention

$U(1) = W(2, c)$. Will get a', d_1 st $a', a' + d_1$ are same col.
Rewrite: $a' = (a' - d_1^2) + d_1^2$. Let $a = a' - d_1^2$.
$a + d_1^2 = a' \qquad a + d_1^2 + d_1 = a' + d_1$
So they have the same color.

If a is that col, have i. If a is diff col, have ii.

There is one thing wrong with this proof. Can you tell?

What if $a' - d_1^2 < 0$? Then $a < 0$.
Can you fix this?

Fix: $U(1) = W(2; c)^2 + W(2; c)$. Do the above in $W(2; c)$ part.

Convention We ignore this issue since we know how to fix it.

Hence our bds are a byte lower than bds in real proof.

The bounds are so big that we don't care.

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GOTO WHITE BOARD

Poly Van Der Warden's (PVDW) Theorem: $\mathrm{PVDW}(x^2, x)$

Exposition by William Gasarch

January 23, 2025

We Begin Proof of $\mathrm{PVDW}(x^2, x)$

Thm For all $c \in \mathbb{N}$ there exists $W = W(x, x^2; c)$ st, for all $\mathrm{COL}\colon [W] \rightarrow [c]$, there exists a, d st

$$a, a + d, a + d^2 \text{ are same col.}$$

We Prove a Lemma Which Implies Theorem

Want:

Thm For all $c \in \mathbb{N}$ there exists $W = W(x, x^2; c)$ st for all $\mathrm{COL}\colon [W] \to [c]$,
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Think about what the lemma will be with your neighbor.

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Lemma proves Theorem by taking $r = c$. Second part can't happen, so first part does.

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GOTO WHITE BOARD

A Powerful Notation and a General Approach

Exposition by William Gasarch

January 23, 2025

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Proofs used $\mathrm{PVDW}(x^2 - \square x \ldots, x^2, \ldots, x^2 + \square x)$ for Base and Ind.

Key There are two lead coefficients and they are for quadratic-degree 2 and linear-degree 1. We will denote this $(1,1)$: 1 quad lead coeff, 1 linear lead coeffs.

Associate to Each Set of Poly's an Index

Notation Let P be a finite subset of $\mathbb{Z}[x]$ such that $(\forall p \in P)[p(0) = 0]$.
Assume the max degree of a poly is d.
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$\{x^4, 2x^4 + \square x^3, x^2, 2x^2, 100x^2, x, 100000x)$ has index $(2, 0, 3, 2)$.

A Powerful Notation

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But what about $\mathrm{PVDW}(1,0)$? That was proven by VDW.

Can we Express VDW in our Powerful Notation?

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Notation Let N^+ be $N \cup \{\omega\}$.
Let $n_d, \ldots, n_1 \in \mathbb{N}^+$ is defined in the obvious way.

What Did We Prove?

Our proof of $\mathrm{PVDW}(x^2)$ has all the ideas to prove $\mathrm{PVDW}(\omega) \implies \mathrm{PVDW}(1,0)$.

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Actual Proof of Poly VDW Theorem

Poly VDW thm proven by ind on the indexes of sets. Ordering:

$$(1) \prec (2) \prec \cdots \prec (\omega) \prec (1,0) \prec (1,1) \prec \cdots \prec (1,\omega)$$

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This is an ω^ω ind. **Contrast** VDW was a ω^2 ind.
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We do this in two parts.

1. Let $0 \le i \le d$. Let $n_d, \ldots, n_i \in \mathbb{N}^+$ with $n_i \in \mathbb{N}$.

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2. $\mathrm{PVDW}(\omega, \ldots, \omega) \implies \mathrm{PVDW}(1, 0, \ldots, 0)$.
   d ω's in the 1st part; d 0's in the 2nd part.

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3. Are better bounds known? See next slide.

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- Proof is elementary. Can be in this class but won't.
- Bounds still large. Not able to write down.
- Proof badly needs someone to write it up better.

A False Prediction

In 1999 there were two thoughts in the air

1. $\mathrm{PVDW}(\vec{n})$ is not prim rec and a **logician** will prove this deep result. Perhaps like the Large Ramsey Numbers (1977) though not that big.
2. $\mathrm{PVDW}(\vec{n})$ is surely prim rec and a **combinatorist** will prove this perhaps with a clever elementary technique.

The above dichotomy is false. The Poly VDW theorem is just not that well known, even now. So there were no **thoughts in the air**. (More on that Later.)
Even so, are there better bounds? VOTE: BETTER BOUNDS KNOWN, BETTER BOUNDS UNKNOWN.

Logician (Shelah) proved $\mathrm{PVDW}(\vec{n})$ prim rec: clever!

- Proof is elementary. Can be in this class but won't.
- Bounds still large. Not able to write down.
- Proof badly needs someone to write it up better.
- Bill- remember to tell them how you learned of Shelah's result.

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So we can obtain a proof of VDW that you can write down nicely.

1. The proof really is the proof I already showed you.
2. While one COULD obtain a clean proof of VDW nobody has bothered writing this up (except me).