Primitive Recursive Function and Ramsey Theory

Exposition by William Gasarch-U of MD

Bounds on a-ary Ramsey Numbers

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We need a way to express very fast growing functions.

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4. $g_1(x_1, \ldots, x_k)$, …, $g_n(x_1, \ldots, x_k)$,$h(x_1, \ldots, x_n)$ PR $\implies$

$$f(x_1, \ldots, x_k) = h(g_1(x_1, \ldots, x_k), \ldots, g_n(x_1, \ldots, x_k)) \text{ is PR}$$

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5. $h(x_1, \ldots, x_{n+1})$ and $g(x_1, \ldots, x_{n-1})$ PR $\Longrightarrow$

$$\begin{aligned} f(x_1, \ldots, x_{n-1}, 0) &= g(x_1, \ldots, x_{n-1}) \\ f(x_1, \ldots, x_{n-1}, m+1) &= h(x_1, \ldots, x_{n-1}, m, f(x_1, \ldots, x_{n-1}, m)) \end{aligned}$$

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The PR functions can be put in a hierarchy depending on how many times the recursion rule is used to build up to the function.

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Used Rec Rule five times.
What should we call this? Discuss

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Its been called WOWER.

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f_6 and beyond have no name.

Levels

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Note One can show that any finite number of exponentials is in PR_3.

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- Is the function $f(a,k) = R_a(k)$ PR?

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Yes. We will see a contrived one on the next slide.

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Let $f_1, f_2, \ldots$ be all of the PR functions.

$$F(x) = f_x(x) + 1$$

is computable but not a PR function.

A “Natural” non PR Function

Def **Ackerman’s function** is the function defined by

$$\begin{aligned} A(0,y) &= y+1 \\ A(x+1,0) &= A(x,1) \\ A(x+1,y+1) &= A(x,A(x+1,y)) \end{aligned}$$

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1. A is obviously computable.
2. A grows faster than any PR function.
3. Since A is defined using a recursion which involves applying the function to itself there is no obvious way to take the definition and make it PR. Not a proof, an intuition.

Ackerman's Function is Natural: Security

https://www.ackermansecurity.com/

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They are called Ackerman Security since they claim that Burglar would have to take time Ackerman(n) to break in.

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So $nA^{-1}(n, n)$ is the exact upper and lower bound!

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1. Goodstein Sequences (next slide packet).
2. Finite Version of Kruskal's Tree Theorem (next next slide packet).