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Some Bounds for the Ramsey–Paris–Harrington Numbers

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It has recently been discovered that a certain variant of Ramsey's theorem cannot be proved in first-order Peano arithmetic although it is in fact a true theorem. In this paper we give some bounds for the "Ramsey–Paris–Harrington numbers" associated with this variant of Ramsey's theorem, involving coloring of pairs. In the course of the investigation we also study certain weaker and stronger partition relations.

1. INTRODUCTION AND NOTATION

We first introduce some appropriate notation. Lower case variables will always denote positive integers, while upper case variables will denote finite sets of positive integers (except when clear from context). We let $|X|$ denote the cardinality of X, $\min X$ the minimum element of X, $[a, b]$ the interval $\{x \mid a \leqslant x \leqslant b\}$, and $[a]$ the interval $[1, a]$. Let $\log x$ denote the logarithm of x to base 2. Given a map F we let $F''Z = \{F(z) \mid z \in Z\}$. Let $F^{(y)}$ denote the yth iterate of F, that is, $F^{(0)}(x) = x$, $F^{(y+1)}(x) = F(F^{(y)}(x))$. Finally, let $[X]^e = \{Y \mid Y \subseteq X \text{ and } |Y| = e\}$.

We now introduce notation generalizing the customary partition calculus. For each $i = 1, 2,..., c$, let a_i be a positive integer or the symbol $*$. Define

$$X \to (a_1, ..., a_c)^e$$

to mean that for any map $F\colon [X]^e \to [c]$ there exists $Y \subseteq X$ and $i \in [c]$ such that $F''[Y]^e = \{i\}$ and

$$|Y| \geqslant a_i \qquad \text{if} \quad a_i \text{ is an integer,}$$

$$\text{and} \left.\begin{matrix} |Y| \geqslant \min Y \\ |Y| > e \end{matrix}\right\} \qquad \text{if} \quad a_i \text{ is } *.$$

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In this context we will often refer to the elements of $[c]$ as *colors* and to F as a c-coloring of $[X]^e$. The set Y is called *homogeneous* since $|F''[Y]^e| = 1$, and *relatively large* when $|Y| \geqslant \min Y$. As usual, if $a_1 = a_2 = \cdots = a_c = a$, we write $X \to (a)^e_c$ for $X \to (a_1,..., a_c)^e$. As we will have occasion to use the ordinary Ramsey function, we define $r(m, n) = \mu p([p] \to (m, n)^2)$.

It is clear that for fixed integers a, e, c the relation $X \to (a)^e_c$ depends only on the cardinality of X. However, $X \to (*)^e_c$ is sensitive to the particular elements in X. The classical Ramsey's theorem states that for all integers a, e, c there exists an x such that $[x] \to (a)^e_c$ (usually written $x \to (a)^e_c$). This theorem is provable from the traditional first-order Peano axioms of arithmetic (PA). In April 1977, Paris discovered that certain combinatorial statements akin to Ramsey's theorem are true but *cannot* be proved from the Peano axioms [7]. Later Harrington, using ideas of Kirby and Paris [4], showed that the statement

$$\forall e \, \forall k \, \forall c \, \exists n \qquad [k, n] \to (*)^e_c \tag{$*$}$$

is also an example of such a statement. From one viewpoint it can be said that the reason for the unprovability of $(*)$ is the fact that the function $R^e_c(k) = \mu n([k, n] \to (*)^e_c)$ grows too rapidly for the axioms of Peano arithmetic to keep pace: If $g(x)$ is any function which PA can prove to be total recursive, then there exists a number e such that $g(x) < R^e_2(x)$ for all sufficiently large x (see [8]). Since R is recursive it follows that PA cannot prove that the diagonal function $R^x_2(x)$ is total (i.e., defined for all x), and *a fortiori* PA cannot prove $(*)$.

It is not true, however, that $(*)$ is *very far* out of the reach of Peano's axioms. In fact for any *fixed* exponent e the following statement *can* be proved in PA:

$$\forall k \, \forall c \, \exists n \qquad ([k, n] \to (*)^e_c). \tag{$*$e}$$

(Cf. Paris and Harrington [8]. Having a separate proof of each instance ($*$e) (infinitely many proofs in all) is not the same as having one single proof of $(*)$. This illustrates the fact that PA is ω-incomplete.) Thus for any fixed exponent e, PA *can* prove that the function $f(k, c) = R^e_c(k)$ is total, whence f does not exhibit quite the same phenomenal growth rate as R itself.

In this paper we concentrate on the function R^2, i.e., Ramsey–Paris–Harrington numbers for partitions of exponent two. In Section 2 we state in the simplest terms the main conclusions of the paper. Section 3 contains further discussion of the results of the paper and mentions results obtained by other authors. In Section 4 we give the proofs. In most cases the results proved in Section 4 are stronger than the versions stated in Section 2. In particular we obtain bounds for certain weaker and stronger partition relations as well.

2. Main Results

Let $R_c(k) = R_c^2(k)$, or in other words,

$$R_c(k) = \mu n([k, n] \to (*)_c^2).$$

Let $R(k) = R_2(k)$. We obtain the following values and bounds for R and R_c.

Theorem 1.

(i) $R(1) = 6$.

(ii) $R(2) = 8$.

(iii) $R(3) = 13$.

(iv) $R(4) \leqslant 687$.

Theorem 2. (i) *There exists* $c > 0$ *such that* $(c\sqrt{k}/\log k)^{2^{k/2}} < R(k)$ *for all sufficiently large* k.

(ii) $R(k) < 2^{k^{2k}}$ *for all* $k \geqslant 2$.

Theorem 3. *Define two sequences of primitive recursive functions as follows*:

$$L_0(k) = k + 1 \qquad L_n(k) = L_{n-1}^{(k-1)}(k) \qquad \text{for} \quad n \geqslant 1,$$
$$U_2(k) = 2^{k^{2k}} \qquad U_3(k) = U_2^{(6k-11)}(k)$$
$$U_n(k) = U_{n-1}^{(n(k-1))}(k) \qquad \text{for} \quad n \geqslant 4.$$

Then

(i) $L_c(k) \leqslant R_c(k)$ *for* $k \geqslant 3, c \geqslant 1$,

(ii) $R_c(k) \leqslant U_c(k)$ *for* $k \geqslant 3, 2 \leqslant c \leqslant k$.

Corollary 4. (i) *For each primitive recursive function* $g(x)$ *there exists a* c *such that* $g(k) \leqslant R_c(k)$ *for all* k.

(ii) *For each* c *there exists a primitive recursive function* $g(x)$ *such that* $R_c(k) \leqslant g(k)$ *for all* k.

3. Remarks

Theorems 2 and 3 are formulated as simply as possible. In each case the actual proof gives considerably more information than what we have stated above. In particular each of the stated lower bounds is in fact a lower bound for a weaker partition relation (cf. Theorems 5, 7, 8) while each of the upper bounds is a simplification of a somewhat sharper upper bound which is more complicated to express and hence less perspicuous (cf. Theorems 6, 9, 10).

Note that $L_c(k)$ and $U_c(k)$, considered as functions of two variables, are simply variants of Ackermann's generalized exponential function. For example, for $k \geqslant 3$ we have $L_2(k) \geqslant 2^k$, $L_3(k) \geqslant 2^{2^{\cdots^2}}$, a stack of k twos, and so forth. We can summarize Theorem 3 as saying simply that $R_c(k)$, as a function of two variables, grows as fast as Ackermann's function. Thus Corollary 4 is an immediate consequence of Theorem 3 by well-known results of mathematical logic. It follows of course that $R_c(k)$, as a function of two variables, has no primitive recursive upper bound.

A further consequence is that $R_2^3(k)$ also grows essentially as fast as Ackermann's function and has no primitive recursive upper bound. Indeed, suppose $k \to (3)_c^2$ and let $I = [k, R_c(k) - 1]$. If $F\colon [I]^2 \to [c]$ refutes $I \to (*)_c^2$, then we get a refutation of $I \to (*)_2^3$ by defining for $X \in [I]^3$

$$G(X) = \begin{cases} 1 & \text{if } X \text{ is homogeneous for } F, \\ 2 & \text{otherwise.} \end{cases}$$

Therefore $R_c(k) \leqslant R_2^3(k)$. It would be interesting to know whether $R_2^3(k) \geqslant R_k(k)$. We remark that the class of primitive recursive functions (as well as Ackermann's function) form a small subset of the class mentioned earlier of all recursive functions which *PA* can prove to be total.

A number of authors have obtained results similar if not equivalent to our Corollary 4(i) (cf. Paris and Harrington [8], Solovay [9], and Joel Spencer, personal communication), but no results as sharp as Theorem 3 have previously been announced. A slightly weaker upper bound for $R(k)$ was obtained earlier in a series of two manuscripts by Máté [5] and [6]. He showed roughly that $R(k) \leqslant (12k)^{(k-2)!2!3!\ldots(k-2)!}$.

Benda [1] has independently obtained upper bounds very similar to our Theorem 2(ii) for a slightly different formulation of the partition relation. Following [8] define

$$n \underset{*}{\longrightarrow} (k)_c^e$$

to mean that for any c-coloring of $[0, n-1]^e$ there exists a relatively large homogeneous set of size $\geqslant k$. Let $r^*(k) = \mu n(n \to_* (k)_2^2)$. Then $r^*(k) < R(k)$ for $k \geqslant 3$. Benda independently arrived at an argument very similar to our proof of Theorem 6 to obtain an upper bound b_k for $r^*(k)$ expressed in terms of iterated ordinary Ramsey numbers. His b_k is conceptually the same as our bound n obtained in Theorem 6.

4. Proofs

Proof of Theorem 1. The lower bounds in (i), (ii), and (iii) are verified by noting that none of the colorings in Fig. 1 contains a relatively large homogeneous set. (Lines join red pairs, no lines join green pairs.)

We now derive the upper bounds.

(i) $R(1) \leqslant 6$. Let $[1, 6]^2$ be colored red and green. The usual proof that $[1, 6] \rightarrow (3)_2^2$ can easily be enhanced to show that there must be at least *two* homogeneous triangles. One of these must intersect $\{1, 2, 3\}$ and hence be relatively large.

(ii) $R(2) \leqslant 8$. Let $[2, 8]^2$ be colored red and green, and suppose there is no relatively large homogeneous set. We will write "xy is red" to mean that $\{x, y\}$ is assigned the color red under this coloring. Let $R_2 = \{x \neq 2 \mid 2x$ is red$\}$ and $G_2 = \{x \neq 2 \mid 2x$ is green$\}$; and similarly $R_3 = \{x \neq 3 \mid 3x$ is red$\}$, $G_3 = \{x \neq 3 \mid 3x$ is green$\}$. W.l.o.g. $3 \in R_2$. By symmetry, $2 \in R_3$. Note that R_3 must be homogeneous green, since otherwise there exist $x, y \in R_3$ such that $\{3, x, y\}$ is relatively large and homogeneous red. Similarly R_2 is homogeneous green while G_2 and G_3 are homogeneous red. Since $2 \in R_3$, $|R_3| < 3$. Since $3 \in R_2$, $|R_2| < 3$. Let $a = \min G_3$. Then $|G_3| < a$. Since $7 = |[2, 8]| = |\{3\} \cup R_3 \cup G_3| \leqslant 1 + 2 + (a - 1)$ we must have $a \geqslant 5$. It follows that $4 \notin G_3$, so $4 \in R_3$. Similarly $4 \notin G_2$, so $4 \in R_2$. But then $\{2, 3, 4\}$ is homogeneous red and relatively large, contradiction.

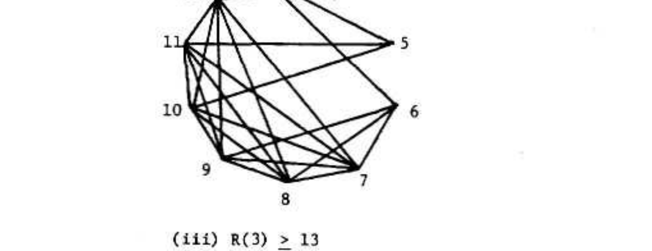

Figure 1

(iii) $R(3) \leqslant 13$. Let $[3, 13]^2$ be colored red and green, and suppose there is no relatively large homogeneous set. Let R_3 and G_3 be as above. W.l.o.g. $4 \in R_3$, so $|R_3| \leqslant 3$. Let $b = \min G_3$, so $|G_3| \leqslant b - 1$. Hence $11 = 1 + |R_3| + |G_3| \leqslant 1 + 3 + (b-1)$, so $b \geqslant 8$. But we cannot have $\{4, 5, 6, 7\} \subseteq R_3$ since $|R_3| \leqslant 3$, so $b \in \{5, 6, 7\}$, contradiction.

(iv) $R(4) \leqslant 687$. Let $[4,687]^2$ be colored red and green, and suppose there is no relatively large homogeneous set. W.l.o.g. 45 is green (i.e., $\{4, 5\}$ is green). Let $b_1 = \mu x$ ($4x$ is red). Define

$$A_1 = \{x > 5 \mid 4x \text{ green, } 5x \text{ red}\},$$

$$A_2 = \{x > 5 \mid 4x \text{ green, } 5x \text{ green}\},$$

$$B_1 = \{x > b_1 \mid 4x \text{ red, } b_1 x \text{ green}\},$$

$$B_2 = \{x > b_1 \mid 4x \text{ red, } b_1 x \text{ red}\}.$$

Let $a_2 = \min A_2$, $b_2 = \min B_2$. Then $[4,687] = \{4, 5, b_1\} \cup A_1 \cup A_2 \cup B_1 \cup B_2$, a disjoint union. (See Fig. 2).

Now $A_1 \nrightarrow (3, 4)^2$ since if $\{x, y, z\} \subseteq A_1$ were homogeneous green then $\{4, x, y, z\}$ would be relatively large and homogeneous green, while if $\{w, x, y, z\} \subseteq A_1$ were homogeneous red then $\{5, w, x, y, z\}$ would be relatively large and homogeneous red. Since $9 \to (3, 4)^2$, we have $|A_1| \leqslant 8$. Now A_2 must be homogeneous red since otherwise there exist $x, y \in A_2$ such that $\{4, 5, x, y\}$ is relatively large and homogeneous green. Therefore $|A_2| < a_2$. Similarly,

$$B_1 \nrightarrow (3, b_1 - 1)^2 \tag{1}$$

and $|B_2| < b_2$. We have

$$\begin{aligned} 684 = |[4,687]| &= 3 + |A_1| + |A_2| + |B_1| + |B_2| \\ &\leqslant 3 + 8 + (a_2 - 1) + |B_1| + (b_2 - 1) \end{aligned}$$

so

$$675 \leqslant a_2 + b_2 + |B_1|. \tag{2}$$

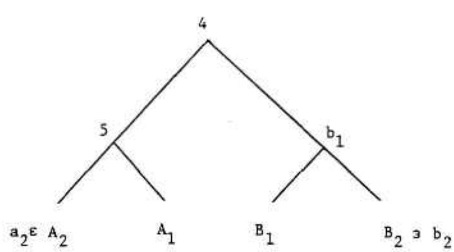

FIGURE 2

We also have

$$b_1 \leqslant 6 + |A_1| + |A_2| \leqslant 13 + a_2 \tag{3}$$

$$\min\{a_2, b_1\} \leqslant 6 + |A_1| \leqslant 14. \tag{4}$$

We now consider three cases: $b_1 \leqslant 14$, $15 \leqslant b_1 \leqslant 26$, and $27 \leqslant b_1$.

Case (I). $b_1 \leqslant 14$. Then by (1), $B_1 \nrightarrow (3, 13)^2$. Since $r(3, 13) \leqslant \binom{14}{2} = 91$, we have $|B_1| \leqslant 90$. Let $c_1 = \min\{a_2, b_2\}$ and $c_2 = \max\{a_2, b_2\}$. Then

$$c_1 \leqslant 7 + |A_1| + |B_1| \leqslant 7 + 8 + 90 = 105$$

and

$$c_2 \leqslant 7 + |A_1| + |B_1| + |C_1| \leqslant 7 + 8 + 90 + 104 = 209$$

where $C_1 = A_2$ if $c_1 = a_2$ and $C_1 = B_2$ if $c_1 = b_2$ (hence $|C_1| \leqslant c_1 - 1$). We conclude from (2) that $675 \leqslant 105 + 209 + 90 = 404$, a contradiction.

Case (II). $15 \leqslant b_1 \leqslant 26$. Then by (1), $B_1 \nrightarrow (3, 25)^2$. In Graver and Yackel [3] it is proved that $r(3, 9) \leqslant 37$. Using the recurrence relation $r(3, n+1) \leqslant r(3, n) + n + 1$, it follows that $r(3, 25) \leqslant 317$. Therefore $|B_1| \leqslant 316$. (An improvement in the estimation of $r(3, 25)$ would yield a corresponding improvement in the bound for $R(4)$. See note added in proof.) Now by (4) we have $a_2 \leqslant 14$, so that

$$\begin{aligned} b_2 &\leqslant 7 + |A_1| + |A_2| + |B_1| \\ &\leqslant 7 + 8 + 13 + 316 = 344. \end{aligned}$$

We conclude from (2) that $675 \leqslant 14 + 344 + 316 = 674$, a contradiction.

Case (III). $b_1 \geqslant 27$. By (4), $a_2 \leqslant 14$. But $|A_2| \leqslant a_2 - 1 \leqslant 13$ and $|A_1| \leqslant 8$, so by (3)

$$27 \leqslant b_1 \leqslant 6 + |A_1| + |A_2| \leqslant 6 + 8 + 13 = 27.$$

Therefore equality holds throughout, and $a_2 = 14$, $|A_1| = 8$, and $|A_2| = 13$. It follows that $A_1 = \{6, 7, \ldots, 13\}$ and $A_2 = \{14, \ldots, 26\}$. Now, since $[6, 9]$ cannot be homogeneous red (else $[5, 9]$ would be), let $\{p, q\} \in [6, 9]^2$ be colored green. Since $\{p\} \cup (R_p \cap A_2)$ is homogeneous red, we must have $|R_p \cap A_2| \leqslant p - 2 \leqslant 6$. Consequently $|G_p \cap A_2| \geqslant 13 - 6 = 7$. Also $|G_p \cap A_1| \geqslant 2$ (since $R_p \cap A_1 \nrightarrow (3, 3)^2$ implies $|R_p \cap A_1| \leqslant 5$). Now $|G_p \cap (A_1 \cup A_2)| \geqslant 9$, $q \in G_p \cap (A_1 \cup A_2)$, and $G_p \cap (A_1 \cup A_2)$ must be homogeneous red to avoid forming a green triangle inside $A_1 \cup A_2$. Since $q \leqslant 9$, $G_p \cap (A_1 \cup A_2)$ is a relatively large homogeneous set, contradiction. This completes the proof of Theorem 1. ∎

Theorem 2(i) is a corollary of the following bound for a weaker partition relation.

THEOREM 5. *Given* k, *let* $n_0 = k$, $n_{i+1} = n_i + r(3, n_i) - 1$ *and* $n = n(k) = n_{k/2-1}$. *Then*

(i) $[k, n-1] \nrightarrow (k, *)^2$.

(ii) *There is a positive constant* c *such that* $n(k) > (c\sqrt{k}/\log k)^{2^{k/2}}$ *for all sufficiently large* k.

Proof. (i) Let $I = [k, n-1]$. We must construct a 2-coloring of $[I]^2$ with no size k homogeneous set of color 1 and no relatively large homogeneous set of color 2. For each $i = 0, 1, ..., k/2 - 2$ pick a coloring $F_i: [n_i, n_{i+1} - 1]^2 \rightarrow [2]$ with no homogeneous triangle of color 1 and no size n_i homogeneous set of color 2. This is possible since $|[n_i, n_{i+1} - 1]| = r(3, n_i) - 1$. Define the coloring $F: [I]^2 \rightarrow [2]$ by

$$F(u, v) = \begin{cases} F_i(u, v) & \text{if } n_i \leqslant u < v < n_{i+1} \\ 1 & \text{otherwise.} \end{cases}$$

Now if $X \subseteq I$ is homogeneous for F to color 1, then for each i, $X \cap [n_i, n_{i+1} - 1]$ is homogeneous for F_i to color 1. Hence $|X \cap [n_i, n_{i+1} - 1]| \leqslant 2$ for all i, so $|X| \leqslant 2(k/2 - 1) < k$. On the other hand if $X \subseteq I$ is homogeneous for F to color 2, then $X \subseteq [n_i, n_{i+1} - 1]$ for some i. Consequently X is homogeneous for F_i to color 2, so $|X| < n_i \leqslant \min X$ and X is not relatively large. Thus F is a counterexample to $I \rightarrow (k, *)^2$, as desired.

(ii) According to a theorem of Erdős [2] there is a positive constant a such that for all sufficiently large m, $r(3, m) \geqslant am^2/(\log m)^2$. Let $b = a/(\log k)^2$. We may assume $b \leqslant 1$. We show inductively for $i = 0, 1, ..., k/2 - 1$ that

$$n_i \geqslant k^{2^i} b^{2^i - 1}/4^{2^i - i - 1}.$$

For $i = 0$ we have $n_0 = k = k^1 b^0/4^0$, as claimed. Now assuming it holds for i, we have

$$\begin{aligned} n_{i+1} &\geqslant r(3, (k^{2^i} b^{2^i - 1}/4^{2^i - i - 1})) \\ &\geqslant a(k^{2^i})^2 (b^{2^i - 1})^2/(4^{2^i - i - 1})^2 (\log k^{2^i})^2 \\ &= ak^{2^{i+1}} b^{2^{i+1} - 2}/4^{2^{i+1} - 2i - 2} 2^{2i} (\log k)^2 \\ &= (k^{2^{i+1}} b^{2^{i+1} - 2}/4^{2^{i+1} - i - 2})(a/(\log k)^2) \\ &= (k^{2^{i+1}} b^{2^{i+1} - 1})/(4^{2^{i+1} - (i+1) - 1}) \end{aligned}$$

as claimed.

Now let $c = \sqrt{a/4}$. Note that c does not depend on k, and we have for sufficiently large k

$$\begin{aligned} n(k) = n_{k/2-1} &\geqslant (k^{2^{k/2-1}} b^{2^{k/2-1}-1})/(4^{2^{k/2-1}-k/2}) \\ &\geqslant (kb/4)^{2^{k/2-1}} \\ &= (c\sqrt{k}/\log k)^{2^{k/2}}. \quad \blacksquare \end{aligned}$$

Proof of Theorem 2(i). Certainly if $X \subseteq [k, n-1]$ is relatively large then $|X| \geqslant k$. Therefore $[k, n-1] \to (*)^2_2$ implies $[k, n-1] \to (k, *)^2$, so $R(k) \geqslant n(k)$ from Theorem 5. ∎

We note that for sufficiently large k, $c\sqrt{k}/\log k > 2$, so we have

$$2^{2^{k/2}} < R(k)$$

for all sufficiently large k.

Theorem 2(ii) will follow as a corollary of the following somewhat sharper upper bound for $R(k)$ involving iteration of ordinary Ramsey numbers.

THEOREM 6. *Let $k \geqslant 3$ be given. Let Σ be the collection of all binary sequences with at most $(k-2)$ zeros and $(k-2)$ ones. Define the number n_σ for each $\sigma \in \Sigma$ by recursion on the length of σ. Let $n_\varnothing = k+1$. Given n_σ, let*

$$n_{\sigma 0} = n_\sigma + r(k-i, n_\sigma - 1)$$

where i is the number of zeros in $\sigma 0$, and

$$n_{\sigma 1} = n_\sigma + r(k-j, n_\sigma - 1)$$

where j is the number of ones in $\sigma 1$.

Let $n = \max\{n_\sigma \mid \sigma \in \Sigma\}$. Then $R(k) \leqslant n$, that is,

$$[k, n] \to (*)^2_2.$$

Proof. Let $[k, m]^2$ be colored red and green, and suppose there is no relatively large homogeneous set. We will show $m < n_\sigma$ for some $\sigma \in \Sigma$, whence $m < n$. Define $a_0 = k$.

$$a_{i+1} = \mu x(x > a_i \text{ and } \{a_0, ..., a_i, x\} \text{ is homogeneous green}).$$

$$\begin{aligned} A_{i+1} = \{x \mid x > a_{i+1}, &\{a_0, ..., a_i, x\} \text{ is homogeneous green and} \\ &a_{i+1}x \text{ is red}\}. \end{aligned}$$

Define $b_0 = k$, b_{i+1}, B_{i+1} analogously with the colors reversed. Note that since $a_0 = k$, a_{k-1} "doesn't exist" (otherwise $\{a_0, ..., a_{k-1}\}$ would be relatively

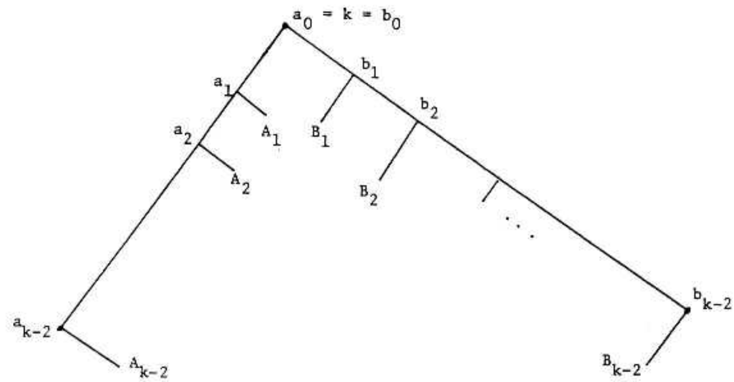

FIGURE 3

large and homogeneous). We will carry out the argument as if all of $\{a_0, ..., a_{k-2}\}$ were defined. The contrary assumption involves only minor notational changes. Note also that $[k, m]$ is equal to the disjoint union $\{k, a_1, a_2, ..., a_{k-2}, b_1, ..., b_{k-2}\} \cup A_1 \cup \cdots \cup A_{k-2} \cup B_1 \cup \cdots \cup B_{k-2}$. See Fig. 3.

We claim that for each $i = 1, 2, ..., k-2$

$$A_i \not\to (k-i, a_i - 1)^2,$$

$$B_i \not\to (k-i, b_i - 1)^2.$$

Indeed, if $\{x_i, ..., x_{k-1}\} \subseteq A_i$ were homogeneous green then $\{a_0, a_1, ..., a_{i-1}, x_i, ..., x_{k-1}\}$ would be relatively large and homogeneous green. If $\{x_1, ..., x_{a_i - 1}\} \subseteq A_i$ were homogeneous red then $\{a_i, x_1, ..., x_{a_i-1}\}$ would be relatively large and homogeneous red. Similarly for B_i with colors reversed. It follows that

$$|A_i| < r(k-i, a_i - 1), \tag{5}$$

$$|B_i| < r(k-i, b_i - 1). \tag{6}$$

Now let $c_1, c_2, ..., c_{2k-3}$ be $a_1, ..., a_{k-2}$, $b_1, ..., b_{k-2}, m+1$, listed in increasing order. (In particular $c_1 = k+1$, $c_{2k-3} = m+1$.) For $i = 1, 2, ..., 2k-4$ define

$$C_i = \begin{cases} A_j & \text{if } c_i = a_j \\ B_j & \text{if } c_i = b_j. \end{cases}$$

Also define a binary sequence σ of length $2k-4$ so that

$$\sigma(i) = \begin{cases} 0 & \text{if } c_i = a_j \text{ for some } j \\ 1 & \text{if } c_i = b_j \text{ for some } j \end{cases}$$

for $i = 1, \ldots, 2k-4$. (Formally, a binary sequence is a function from some $[x]$ to $\{0, 1\}$.) Clearly $\sigma \in \Sigma$. We claim that $m < n_\sigma$. To prove this we show inductively that

$$c_i \leqslant k + i + \sum_{1 \leqslant j < i} |C_j| \leqslant n_{\sigma \restriction [i-1]} \tag{7}$$

for $i = 1, 2, \ldots, 2k-3$, where $\sigma \restriction [i-1]$ denotes the restriction of σ to $[i-1]$. We have $c_1 = k+1 = k+1+\sum_{1 \leqslant j < 1} |C_j|$ and $n_{\sigma \restriction [0]} = n_\varnothing = k+1$, so (7) holds for $i = 1$. For $i > 1$ the left-hand inequality in (7) is clear from the definition of the C_j's. For the right-hand inequality, consider the case $\sigma(i) = 0$. Then $c_i = a_{i'}$ and $C_i = A_{i'}$ for some i' (i' is the number of zeros in $\sigma \restriction [i]$), and we have

$$\begin{aligned} n_{\sigma \restriction [i]} &= n_{\sigma \restriction [i-1]} + r(k-i', n_{\sigma \restriction [i-1]} - 1) && \text{by definition} \\ &\geqslant k + i + \sum_{1 \leqslant j < i} |C_j| + r(k-i', a_{i'} - 1) && \text{since } n_{\sigma \restriction [i-1]} \geqslant c_i = a_{i'} \\ &\geqslant k + i + \sum_{1 \leqslant j < i} |C_j| + |A_{i'}| + 1 && \text{by (5)} \\ &= k + (i+1) + \sum_{1 \leqslant j \leqslant i} |C_j| && \text{as required.} \end{aligned}$$

The case $\sigma(i) = 1$ is analogous. This proves (7).

We conclude that $c_{2k-3} \leqslant n_{\sigma \restriction [2k-4]} = n_\sigma$. But $c_{2k-3} = m+1$, so $m < n_\sigma$. This concludes the proof of Theorem 6. ∎

We note that Theorem 6 yields an upper bound for $R(5)$ on the order of 3×10^{17} by actually calculating upper bounds for all the n_σ's.

Proof of Theorem 2(ii). We prove that in Theorem 6

$$n_\sigma \leqslant 2(2(k+1)^{(k-2)!})^{(k-2)!} \qquad \text{for all} \quad \sigma \in \Sigma \tag{8}$$

whence

$$R(k) \leqslant 2(2(k+1)^{(k-2)!})^{(k-2)!}. \tag{9}$$

We use the fact that

$$r(e, s-1) \leqslant \binom{s+e-3}{e-1} \leqslant s^{e-1} - s^{e-2}$$

for $2 \leqslant e \leqslant s$. We have $n_{\varnothing} = k+1$;

$$\begin{aligned} n_{\sigma 0} &= n_\sigma + r(k-i, n_\sigma - 1) \\ &\leqslant n_\sigma + n_\sigma^{(k-i-1)} - n_\sigma^{(k-i-2)} \\ &\leqslant n_\sigma^{(k-i-1)} \qquad \text{for} \quad i < k-2 \end{aligned}$$

or

$$n_{\sigma 0} \leqslant 2n_\sigma^{(k-i-1)} = 2n_\sigma \qquad \text{for} \quad i = k-2.$$

Similarly, $n_{\sigma 1} \leqslant n_\sigma^{(k-j-1)}$ if $j < k-2$, and $n_{\sigma 1} \leqslant 2n_\sigma$ if $j = k-2$. It follows that $n_\sigma \leqslant 2(2(k+1)^{\nu_1 \nu_2 \cdots \nu_r})^{\nu_{r+1} \cdots \nu_s}$ for each $\sigma \in \Sigma$, where $\prod_{i=1}^{s} \gamma_i \leqslant (k-2)!^2$ and $\prod_{i=r+1}^{s} \gamma_i \leqslant (k-2)!$. The bound (8) follows.

We now have

$$\begin{aligned} R(k) &\leqslant 2(2(k+1)^{(k-2)!})^{(k-2)!} \\ &\leqslant 2^{(k-1)!^2} < 2^{k^{2k}}. \quad \blacksquare \end{aligned}$$

Theorem 3(i) is an immediate corollary of Theorem 7 which shows that in fact $L_c(k)$ is a lower bound for a weaker partition relation. Given a coloring of $[X]^2$, a subset $Y \subseteq X$ is said to be *path-homogeneous* if and only if every pair of *consecutive* elements of Y receives the same color. Clearly this is weaker than being homogeneous. Let $\mathbf{R}_c(k)$ denote the last n such that for every c-coloring of $[k, n]^2$ there exists a relatively large *path*-homogeneous subset of $[k, n]$. Then $\mathbf{R}_c(k) \leqslant R_c(k)$ and we have

THEOREM 7. *For* $c \geqslant 1$, $k \geqslant 3$, $L_c(k) \leqslant \mathbf{R}_c(k)$.

Proof. We give a direct proof. Given c, k, let $I = [k, L_c(k) - 1]$. We claim the following c-coloring of $[I]^2$ contains no relatively large path-homogeneous set:

$$F(x, y) = \max\{n \mid \exists i\, x < L_n^{(i)}(k) \leqslant y\}.$$

Indeed, suppose $X = \{x_1, x_2, \ldots, x_m\} \subseteq I$ is path-homogeneous for F with $x_1 < x_2 < \cdots < x_m$. We must show $m < x_1$.

We know that for some $n \in [0, c-1]$ and for all $i \in [m-1]$, $F(x_i, x_{i+1}) = n$. This means there exist integers $r_1 < r_2 < \cdots < r_{m-1}$ such that $x_i < L_n^{(r_i)}(k) \leqslant x_{i+1}$ and for all integers r, either $L_{n+1}^{(r)}(k) \leqslant x_i$ or $x_{i+1} < L_{n+1}^{(r)}(k)$. Let r be maximal such that $L_{n+1}^{(r)}(k) \leqslant x_1$ and let $s = L_{n+1}^{(r)}(k)$. It follows that $x_m < L_{n+1}^{(r+1)}(k) = L_{n+1}(s) = L_n^{(s-1)}(s)$. On the other hand using the monotonicity of L_n for arguments $\geqslant 3$, we establish inductively that $L_n^{(i)}(s) \leqslant x_{i+1}$ for $i = 0, 1, 2, \ldots, m-1$. Thus $L_n^{(m-1)}(s) \leqslant x_m < L_n^{(s-1)}(s)$, so $m < s \leqslant x_1$ and we are done. $\blacksquare$

We note that it is also possible to establish Theorem 7 inductively by showing that in fact for each c,

$$\mathbf{R}_{c-1}^{(k-1)}(k) \leqslant \mathbf{R}_c(k).$$

This gives a slightly stronger result, assuming, as is likely, that $L_{c-1}(k) < \mathbf{R}_{c-1}(k)$. The same sort of argument will establish that $R_{c-1}^{(k-1)}(k) \leqslant R_c(k)$. In fact an even stronger result will be proved in Theorem 8. Thus using Theorem 2(i) we could have defined the sequence of L functions starting with $L_2(k) = 2^{2^{k/2}}$.

We now turn our attention to a more general case of the Ramsey–Paris–Harrington partition relation. We define

$$R_c(k; m) = \mu n([k, n] \to (m, \bar{*})_c^2)$$

where $\bar{*}$ denotes a sequence of $c-1$ stars. In other words the homogeneous set is required to have size $\geqslant m$ if it is of color 1 and to be relatively large (and of size $\geqslant 3$) if it is of a color greater than one. As a special case we have $R_c(k) = R_{c+1}(k; 2)$. Other special cases are $R_1(k; m) = k + m - 1$ and $R_c(k; 1) = k$. Theorem 5 expresses the fact that for some $c > 0$ eventually $(c\sqrt{k}/\log k)^{2^{k/2}} < R_2(k; k)$.

We remark that for any k, $m \leqslant h$

$$R_c(k; m) \leqslant R_c(h).$$

This holds since $[h, n] \to (*)_c^2$ implies $[k, n] \to (m, \bar{*})_c^2$ whenever $k, m \leqslant h$.

The following theorem gives the basis for an alternative inductive proof (which we shall not spell out) of Theorem 3(i).

Theorem 8. *For $m, c \geqslant 1$ and $k \geqslant 3$*

(i) $R_c^{(m-1)}(k) \leqslant R_{c+1}(k; m)$,

(ii) $R_c^{(k-1)}(k) \leqslant R_{c+1}(k)$.

Proof. (i) For each $i = 1, 2, \ldots, m-1$ let $I_i = [R_c^{(i-1)}(k), R_c^{(i)}(k) - 1]$. Let F_i be a c-coloring of $[I_i]^2$ with no relatively large homogeneous set. Define the $(c+1)$-coloring F on $[k, R_c^{(m-1)}(k) - 1]$ by

$$F(a, b) = \begin{cases} F_i(a, b) + 1 & \text{if } \ a, b \in I_i \text{ some } i \\ 1 & \text{otherwise.} \end{cases}$$

If X is homogeneous for F to color 1, then $|X \cap I_i| \leqslant 1$ for each i, so $|X| \leqslant m - 1$. If X is homogeneous for F to a color greater than 1, then $X \subseteq I_i$ for some i. Hence X is homogeneous for F_i and thus not relatively large. Thus $[k, R_c^{(m-1)}(k) - 1] \nrightarrow (m, \bar{*})_{c+1}^2$.

(ii) By the remark immediately preceding this theorem, we have $R_{c+1}(k) \geqslant R_{c+1}(k; k) \geqslant R_c^{(k-1)}(k)$. ∎

The following theorem gives the key inductive relationship to be used in calculating upper bounds for $R_c(k; m)$ and hence for $R_c(k)$.

THEOREM 9. *Let $c \geqslant 1$ be given and suppose*

$$R_c(k; m) \leqslant g(k, m) \qquad \textit{for all} \quad k, m \geqslant 1.$$

Define

$$f(k, 1) = k$$
$$f(k, m+1) = g(f(k, m)+1, r(m+1, mc(f(k, m)-2)+1)).$$

Then

$$R_{c+1}(k; m) \leqslant f(k, m) \qquad \textit{for all} \quad k, m \geqslant 1.$$

Proof. Fix k and proceed by induction on m. By the special case noted above $R_{c+1}(k; 1) = k = f(k, 1)$, so the conclusion holds for $m = 1$. Now assume inductively that $R_{c+1}(k; m) \leqslant f(k, m)$ and we wish to prove $R_{c+1}(k; m+1) \leqslant f(k, m+1)$. Let $P\colon [k, f(k, m+1)]^2 \to [c+1]$ be given. If there exists a relatively large $X \subseteq [k, f(k, m)]$ which is homogeneous for P to some color $d \geqslant 2$, we are done. So assume there is none, and by the induction hypothesis find a set of m elements $a_1 < a_2 < \cdots < a_m$ in $[k, f(k, m)]$ which is homogeneous for P to color 1.

Let $I = [f(k, m)+1, f(k, m+1)]$. If for some $a \in I$ we have $P(a_i, a) = 1$ for all $i \in [m]$, then again we are done, for $\{a_1, a_2, ..., a_m, a\}$ will be a size $m+1$ set homogeneous for P to color 1. So assume no such $a \in I$ exists and express I as a disjoint union

$$I = \bigcup \{A_{ij} \mid 1 \leqslant i \leqslant m, 2 \leqslant j \leqslant c+1\}$$

so that $P(a_i, a) = j$ for all $a \in A_{ij}$.

We now alter the $(c+1)$-coloring P on I somewhat to obtain a c-coloring $Q\colon [I]^2 \to [c]$ by stipulating

$$\begin{aligned} Q(a, b) &= P(a, b) && \text{if} \quad a, b \in A_{ij} \text{ and } P(a, b) < j \\ &= P(a, b) - 1 && \text{if} \quad a, b \in A_{ij} \text{ and } P(a, b) > j \\ &= 1 && \text{otherwise.} \end{aligned}$$

Thus all lines between points in different A_{ij}'s are changed to color 1. Within

A_{ij}, lines of color $<j$ are left fixed, lines of color j are changed to color 1, and lines of color $>j$ are decreased one color.

By the defining equation for $f(k, m+1)$ one of the following two cases must occur.

Case 1. There exists $X \subseteq I$ which is relatively large and homogeneous for Q to some color $d \geqslant 2$. Then $X \subseteq A_{ij}$ for some i, j. Since we cannot have $P(a, b) < j < P(r, s)$ and $P(a, b) = P(r, s) - 1$ for any $a, b, r, s \in A_{ij}$, we must have either $d = Q(a, b) = P(a, b)$ for all $\{a, b\} \in [X]^2$ or $d = Q(a, b) = P(a, b) - 1$ for all $\{a, b\} \in [X]^2$. Thus X is homogeneous for P to color either d or $d + 1$, and we are done.

Case 2. There exists $X \subseteq I$ which is homogeneous for Q to color 1 and $|X| \geqslant r(m+1, mc(f(k, m) - 2) + 1)$. In this case define $R: [X]^2 \to [2]$ by

$$R(a, b) = 1 \quad \text{if} \quad P(a, b) = 1$$
$$= 2 \quad \text{if} \quad P(a, b) > 1.$$

By the definition of $r(x, y)$ one of the following two subcases must occur.

Subcase (i). There exists $Y \subseteq X$ which is homogeneous for R to color 1 and $|Y| \geqslant m + 1$. Then Y is also homogeneous for P to color 1, and we are done.

Subcase (ii). There exists $Y \subseteq X$ which is homogeneous for R to color 2 and $|Y| \geqslant mc(f(k, m) - 2) + 1)$. By the pigeonhole principle $|Y \cap A_{ij}| \geqslant f(k, m) - 1 \geqslant a_i - 1$ for some A_{ij}, since there are at most mc different A_{ij}'s. We have $P(a, b) = j$ for all $a, b \in Y \cap A_{ij}$, since $Q(a, b) = 1$ and $P(a, b) > 1$. Therefore $(Y \cap A_{ij}) \cup \{a_i\}$ is relatively large and homogeneous for P to color $j \geqslant 2$.

This completes the proof of Theorem 8. ∎

COROLLARY 10. *Define the function $U = U(c, k, m)$ by the equations*

$$U(1, k, m) = k + m - 1,$$
$$U(c+1, k, 1)) = k,$$
$$U(c+1, k, m+1)$$
$$= U(c, U(c+1, k, m) + 1, r(m+1, mc(U(c+1, k, m) - 2) + 1)).$$

Then

$$R_c(k; m) \leqslant U(c, k, m),$$
$$R_c(k) \leqslant U(c+1, k, 2). \quad ∎$$

COROLLARY 11. *For any* $c \geqslant 1, k \geqslant 3$,

$$R_c(k) \leqslant R_c(k+1; c(k-2)+1).$$

Proof. Let $g(k, m) = R_c(k; m)$ and define $f(k, m)$ as in Theorem 8. Then

$$\begin{aligned} R_c(k) &= R_{c+1}(k; 2) \\ &\leqslant f(k, 2) \\ &= R_c(f(k, 1) + 1; r(2, 1 \cdot c \cdot (f(k, 1) - 2) + 1)) \\ &= R_c(k+1; c(k-2)+1). \quad \blacksquare \end{aligned}$$

For the following corollary let $E(x) = x^{3x}$, and given function $f(x)$ let $f^{[y]}$ denote the yth iterate of $f \circ E$, so that $f^{[y+1]}(x) = f(E(f^{[y]}(x)))$. In the proof of the following corollary and in subsequent proofs we will make frequent implicit use of the monotonicity of E, R_c, and U_c. We also use the fact that $E(h) \leqslant U_2(h)$ for all $h \geqslant 1$.

COROLLARY 12. *For* $3 \leqslant k$, $2 \leqslant c \leqslant k$, *and* $1 \leqslant m$

$$R_{c+1}(k; m) \leqslant R_c^{[m-1]}(k).$$

Proof. Let $g(k, m) = R_c(k; m)$ and define $f(k, m)$ as in Theorem 8. We show by induction on m that in fact $f(k, m) \leqslant R_c^{[m-1]}(k)$. For $m = 1$ we have $f(k, 1) = k = R_c^{[0]}(k)$.

Now suppose the corollary holds for a given $m \geqslant 1$ and we wish to prove it for $m+1$. Let $B = r(m+1, mc(f(k, m) - 2) + 1)$ and $h = f(k, m)$. Since $m+1, c, k \leqslant h$ we have

$$B \leqslant r(h, h^3 - 1) \leqslant h^{3h} = E(f(k, m)).$$

Therefore, using the remark preceding Theorem 8 and the monotonicity of R_c and E, we have

$$\begin{aligned} f(k, m+1) &= R_c(h+1; B) \\ &\leqslant R_c(B) \\ &\leqslant R_c(E(f(k, m))) \\ &\leqslant R_c(E(R_c^{[m-1]}(k))) = R_c^{[m]}(k). \quad \blacksquare \end{aligned}$$

LEMMA 13. *For any* $c \geqslant 2$ *and* $k \geqslant 1$

$$U_2(U_c(k)) \leqslant U_c(U_2(k)).$$

Proof. This is trivial for $c = 2$. Assuming it holds for a given $c \geqslant 3$, we have

$$\begin{aligned} U_2(U_{c+1}(k)) &= U_2(U_c^{((c+1)(k-1))}(k)) && \text{definition } U_{c+1} \\ &\leqslant U_c^{((c+1)(k-1))}(U_2(k)) && \text{induction} \\ &\leqslant U_c^{((c+1)(U_2(k)-1))}(U_2(k)) && \text{monotonicity} \\ &= U_{c+1}(U_2(k)). && \text{definition } U_{c+1}. \end{aligned}$$

With trivial modifications the above argument works also for $c = 3$, hence by induction we are done. ∎

Proof of Theorem 3(ii). For $c = 2$ we have $R_2(k) \leqslant 2^{k^{2k}} = U_2(k)$ by Theorem 2(ii). For $c = 3$ we have

$$\begin{aligned} R_3(k) &\leqslant R_3(k+1; 3(k-2)+1) && \text{Corollary 11} \\ &\leqslant R_2^{[3k-6]}(k+1) && \text{Corollary 12} \\ &\leqslant U_2^{(6k-12)}(k+1) && \text{monotonicity} \\ &\leqslant U_2^{(6k-11)}(k) = U_3(k). \end{aligned}$$

Now assume the theorem holds for a given $c \geqslant 3$ and we wish to prove it for $c + 1$. Letting $K = (c+1)(k-2)$ we have

$$\begin{aligned} R_{c+1}(k) &\leqslant R_{c+1}(k+1, K+1) && \text{Corollary 11} \\ &\leqslant R_c^{[K]}(k+1) && \text{Corollary 12} \\ &\leqslant U_c^{(K)}(U_2^{(K)}(k+1)) && \text{monotonicity and Lemma 13} \\ &\leqslant U_c^{(K)}(U_3(K+1)) && \text{since } K \leqslant 6(K+1)-1) \\ &\leqslant U_c^{(K)}(U_3(U_c(k))) && \text{since } K+1 \leqslant U_c(k) \\ &\leqslant U_c^{(K+2)}(k) && \text{monotonicity} \\ &\leqslant U_c^{((c+1)(k-1))}(k) = U_{c+1}(k). \end{aligned}$$ ∎

Note added in proof. Grinstead and Roberts [10] have recently announced that $r(3, 9) = 36$. This enables us to improve Theorem 1(iv) to $R(4) \leqslant 685$.

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