The Infinite 2-ary Can Ramsey Thm

William Gasarch-U of MD

Hungarian Math Comp Problem

From the 1950 "Kürschák/Eötvös Math Competition":

There are 1950 cans of paint. Find an x *such that (1) there are either* x *cans of paint all the same color, or* x *cans of paint that are all different colors and (2) it is possible to have neither* $x+1$ *cans that are all the same nor* $x+1$ *cans that are all different.*

Prove with your neighbor.

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Answer is $x = 45$:

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2) If there are 45 of the same color then DONE

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3) If there are ≤ 44 diff colors and each color appears ≤ 44 times then $\leq 44 * 44 = 1936 < 1950$ cans.

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Answer is $x = 45$:
1) If there are 45 different paint colors DONE
2) If there are 45 of the same color then DONE
3) If there are ≤ 44 diff colors and each color appears ≤ 44 times then $\le 44 * 44 = 1936 < 1950$ cans.
4) CAN have NEITHER 46 the same NOR 46 different:
Color 1st 45 1, 2nd 45 2, ..., 43nd 45 43. You've colored $43 \times 45 = 1935$. Color the rest 44. Have used 44 colors.

Can Ramsey Thm

The Can Ramsey Thm is for any number of colors.

It is named "Can Ramsey" in honor of the paint can problem on the 1950 Kürschák/Eötvös Math Competition

1-ary Ramsey's Thm

Thm: For every $COL : \mathbb{N} \rightarrow [c]$ there is an infinite homog set.

What if the number of colors was **infinite?**

Do not necessarily get a homog set since could color EVERY vertex differently. But then get infinite **rainbow set**.

One-Dim Can Ramsey Thm

Thm: Let V be a countable set. Let $COL : V \to \omega$. Then there exists either an infinite homog set (all the same color) or an infinite rainb set (all diff colors).

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Prove with your neighbor.

Ramsey's Thm For Graphs

Thm: For every $COL : \binom{\mathbb{N}}{2} \rightarrow [c]$ there is an infinite homog set.

What if the number of colors was **infinite?**

Do not necessarily get a homog set since could color EVERY edge differently. But then get infinite **rainbow set**.

Attempt

Conjecture For every $COL : \binom{\mathbb{N}}{2} \rightarrow \omega$ there is an infinite homog set OR an infinite rainb set.
VOTE: TRUE, FALSE, or UNKNOWN TO SCIENCE.

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VOTE: TRUE, FALSE, or UNKNOWN TO SCIENCE.
FALSE:

- $COL(i,j) = \min\{i,j\}$.
- $COL(i,j) = \max\{i,j\}$.

Min-Homog, Max-Homog, Rainbow

Def: Let $COL : \binom{\mathbb{N}}{2} \rightarrow \omega$. Let $V \subseteq \mathbb{N}$. Assume $a < b$ and $c < d$.

- V is *homog* if $COL(a, b) = COL(c, d)$ iff $TRUE$.
- V is *min-homog* if $COL(a, b) = COL(c, d)$ iff $a = c$.
- V is *max-homog* if $COL(a, b) = COL(c, d)$ iff $b = d$.
- V is *rainb* if $COL(a, b) = COL(c, d)$ iff $a = c$ and $b = d$.

Can Ramsey Thm for $\binom{\mathbb{N}}{2}$: For all $COL : \binom{\mathbb{N}}{2} \rightarrow \omega$, there exists an infinite set V such that either V is homog, min-homog, max-homog, or rainb.

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2. Use the **3-ary Ramsey Theorem** to prove the **2-ary Can Ramsey Theorem**.

Our First "Application"

We will do the following

1. Use the **4-ary Ramsey Theorem** to prove the **2-ary Can Ramsey Theorem**.
2. Use the **3-ary Ramsey Theorem** to prove the **2-ary Can Ramsey Theorem**.
3. Use a similar technique from **2-ary Ramsey Theorem** to prove **2-ary Can Ramsey**.

Proof of Can Ramsey Thm for $\binom{\mathbb{N}}{2}$

We are given $COL : \binom{\mathbb{N}}{2} \rightarrow \omega$.
Want to find infinite homog OR min-homog OR max-homog OR rainbow set.

We use COL to define $COL' : \binom{\mathbb{N}}{4} \rightarrow [16]$
We then apply **4-ary Ramsey Theorem.** (an **"Application!"**)

In the slides below $x_1 < x_2 < x_3 < x_4$.
All cases assume negation of prior cases.

Homog always means infinite Homog.

Pairs that begin the same way are same color

1. $COL(x_1, x_2) = COL(x_1, x_3) \rightarrow COL'(x_1 < x_2 < x_3 < x_4) = 1.$
2. $COL(x_1, x_2) = COL(x_1, x_4) \rightarrow COL'(x_1 < x_2 < x_3 < x_4) = 2.$
3. $COL(x_1, x_3) = COL(x_1, x_4) \rightarrow COL'(x_1 < x_2 < x_3 < x_4) = 3.$
4. $COL(x_2, x_3) = COL(x_2, x_4) \rightarrow COL'(x_1 < x_2 < x_3 < x_4) = 4.$

H is homog set,color 1 (rest similar)
$COL'' : H \rightarrow \omega$ is $COL''(x) =$ color of all (x, y) with $x < y \in H$.

Use **1-dim Can Ramsey!**:
Case 1: COL'' has homog set H' then H' homog for COL.
Case 2: COL'' has rainb set H' then H' min-homog for COL.

Pairs that End the same way are same color

1. $COL(x_1, x_3) = COL(x_2, x_3) \rightarrow COL'(x_1 < x_2 < x_3 < x_4) = 5.$
2. $COL(x_1, x_4) = COL(x_2, x_4) \rightarrow COL'(x_1 < x_2 < x_3 < x_4) = 6.$
3. $COL(x_1, x_4) = COL(x_3, x_4) \rightarrow COL'(x_1 < x_2 < x_3 < x_4) = 7.$
4. $COL(x_2, x_4) = COL(x_3, x_4) \rightarrow COL'(x_1 < x_2 < x_3 < x_4) = 8.$

H is homog set,color 5 (rest similar)
$COL'' : H \rightarrow \omega$ is $COL''(y) =$ color of all (x, y) with $x < y \in H$.

Use **1-dim Can Ramsey!**:
Case 1: COL'' has homog set H' then H' homog for COL.
Case 2: COL'' has rainb set H' then H' max-homog for COL.

Easy Homog Cases

1. $COL(x_1, x_2) = COL(x_2, x_3) \Rightarrow COL'(x_1, x_2, x_3, x_4) = 9.$
2. $COL(x_1, x_2) = COL(x_2, x_4) \Rightarrow COL'(x_1, x_2, x_3, x_4) = 10.$
3. $COL(x_1, x_2) = COL(x_3, x_4) \Rightarrow COL'(x_1, x_2, x_3, x_4) = 11.$
4. $COL(x_1, x_3) = COL(x_2, x_4) \Rightarrow COL'(x_1, x_2, x_3, x_4) = 12.$
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6. $COL(x_2, x_3) = COL(x_1, x_4) \Rightarrow COL'(x_1, x_2, x_3, x_4) = 14.$
7. $COL(x_2, x_3) = COL(x_3, x_4) \Rightarrow COL'(x_1, x_2, x_3, x_4) = 15.$

H is homog set,color 9 (rest similar)
For all $w < x < y < z \in H$.

$$COL(w, x) = COL(x, y) = COL(y, z).$$

Other cases, like $COL(w, y) = COL(x, z)$, are similar

That Last Case

If **NONE** of the above cases hold then $COL'(x_1, x_2, x_3, x_4) = 16$.

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Let H be the homogenous set of COL' of color 16.

Then H is a rainbow set for COL. Leave this to the reader, thought it is obvious.

PROS and CONS of Proof

Give me a PRO and a CON of the proof.

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PRO: Each Case easy. Note that Rainbow case was easy.

CON: Lots of Cases. Use of 4-ary hypergraph Ramsey makes finite version have large bounds.
Let $\mathrm{CR}_2(k) =$ least n s.t. $\forall \mathrm{COL}\colon \binom{[n]}{2} \rightarrow \omega$, $\exists H$ of size k such that either H is homog, min-homog, max-homog, or rainb. If finitized, this proof obtains

$$\mathrm{CR}_2(k) \le R_4(k, 16) \le 16^{16^{16^{O(k)}}}$$

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We will give anther proof which only uses 3-ary hypergraph Ramsey.

Def that Will Help Us

Def Let $COL : \binom{\mathbb{N}}{2} \rightarrow \omega$. If c is a color and $v \in \mathbb{N}$ then $\deg_c(v)$ is the number of c-colored edges with v as an endpoint.

Note: $\deg_c(v)$ could be infinite.

Needed Lemma

Lemma Let X be infinite. Let $COL : \binom{X}{2} \rightarrow \omega$. If for every $x \in X$ and $c \in \omega$, $\deg_c(x) \le 1$ then there is an infinite rainb set.

Prove with your Neighbor

Proof

Let M be a MAXIMAL rainb set of X.

$$(\forall y \in X - M)[M \cup \{y\} \text{ is not a rainb set}].$$

We prove M is infinite.

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Must be that:

$$(\exists u \in M, \exists \{a,b\} \in \binom{M}{2})[COL(y,u) = COL(a,b)].$$

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then $COL(y_1, u) = COL(y_2, u)$.

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So have injection from **infinite** $X - M$ to **finite** $M \times \binom{M}{2}$.
Contradiction So M is infinite.

Generalization We'll Need Later

Lemma Let X be infinite. Let $COL : \binom{X}{2} \to \omega$. Let $d \in \omega$. If for every $x \in X$ and $c \in \omega$, $\deg_c(x) \le d$ then there is an infinite rainb set.

Prove on your own.

Can Ramsey Thm for $\mathbb{N}$

Thm: For all $COL : \binom{\mathbb{N}}{2} \rightarrow \omega$ there is either

- an infinite homog set,
- an infinite min-homog set,
- an infinite max-homog set, or
- an infinite rainb set.

Proof of Can Ramsey Thm for Graphs

Given $COL : \binom{\mathbb{N}}{2} \rightarrow \omega$. We use COL to obtain $COL' : \binom{\mathbb{N}}{3} \rightarrow [4]$
We use 3-ary RT. In all of the below $x_1 < x_2 < x_3$.

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Cases 1,2,3 are just like in the prior proof.

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Case 4 Next slide.

Proof of Can Ramsey Thm for Graphs (cont)

Case 4 for all $x_1 < x_2 < x_3$
$COL(x_1, x_2) \neq COL(x_1, x_3)$
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From this can show that, for all x, for all c, $\deg_c(x) \leq 1$.

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From this can show that, for all x, for all c, $\deg_c(x) \leq 1$. By Lemma on last slide there exists $M \subseteq H$ that is an infinite rainb set.

Better Bounds on Can Ramsey

Using 4-ary proof, 16 colors, bound was:

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$$\mathrm{CR}_2(k) \leq 16^{16^{16^{O(k)}}}$$

Using new proof, 3-ary with 4 colors, bound is:

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$$\mathrm{CR}_2(k) \leq 16^{16^{16^{O(k)}}}$$

Using new proof, 3-ary with 4 colors, bound is:

$$\mathrm{CR}_2(k) \leq 4^{4^{O(k^3)}}$$