BILL, RECORD LECTURE!!!!

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Folkman's Theorem

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More generally, we want **all** non-empty sums are the same color.

$b_1 < b_2$ Theorem

Thm $(\forall c)(\exists T = T(c))$ st $\forall\ \mathrm{COL} : [T] \rightarrow [c]\ \exists b_1 < b_2$ st $\mathrm{COL}(b_2) = \mathrm{COL}(b_1 + b_2)$

Let $T = 3c$ (this is prob not optimal).

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Note $b_1 < b_2$ thm follows from Schur, but we wanted elt proof.

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Thm $(\forall c, d)(\exists T = T(c)d)$ st $\forall$ $\mathrm{COL} : [T] \rightarrow [c]$ $\exists b_1 < b_2$ st $\mathrm{COL}(b_2) = \mathrm{COL}(b_1 + b_2)$ AND $b_1 < b_2$ AND $b_1, b_2 \equiv 0 \pmod{d}$.

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WRITE DOWN WHAT $T(c, d)$ MEANS FOR LATER USE.

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Note that these can be different colors.

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Will prove on next slides.

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We later show general case of $b_1 < \cdots < b_n$.

Proof of $b_1 < b_2 < b_3$ Theorem

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U will be in two blocks, both **very large**.

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Note that $d \le \frac{W(k,c)}{k}$.

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Proof of $b_1 < b_2 < b_3$ Theorem

Fix c. U is TBD. Assume there is $\mathrm{COL}\colon [U]\rightarrow[c]$.

U will be in two blocks, both **very large**.

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$(\exists a,d)[a, a+d, \ldots, a+(k-1)d$ same color $]$ a will be b_3.

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Proof of $b_1 < b_2 < b_3$ Theorem (cont)

Recap proof so far

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Upshot Need $b_2' + b_1' \leq k$. Take $k = 2T(c) = 6c$.

Proof of $b_1 < b_2 < b_3$ Theorem (cont)

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Block2 is $W(6c, c)$ **Block1** is $3cW(6c, c)$.

$$U(c) = 3cW(6c, c) + W(6c, c) = (3c+1)W(6c, c).$$

Summarize Proof

$U = 2W(k, c)$ where $k = 2W(dT(c), c)$.

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$b_1 < \cdots < b_n$ Theorem

Thm $(\forall n, c)(\exists U = U(n,c))$ st $\forall\ \mathrm{COL} : [U] \rightarrow [c]$ $\exists b_1 < \cdots < b_n$ st

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$b_1 < \cdots < b_n$ Theorem

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Will prove on next slides.

Proof of $b_1 < \cdots < b_n$ Theorem

Fix c. U is TBD. Assume there $\mathrm{COL}\colon [U] \to [c]$.

Proof of $b_1 < \cdots < b_n$ Theorem

Fix c. U is TBD. Assume there $\mathrm{COL}\colon [U] \to [c]$.

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Proof on Next Slide.

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