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Probabilistic Method Proof of Turan's Theorem

Exposition by William Gasarch

The Prob Method

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3) We will use the Prob Method to Proof Turan's Theorem.

Turan's Theorem

Theorem If $G = (V, E)$ is a graph, $|V| = n$, and $|E| = e$, then G has an ind set of size at least

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more easily using Probability, but first need a lemma. The proof

we give is due to Ravi Boppana and appears in the Alon-Spencer book on *The Probabilistic Method*

Lemma

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Proof: Try to count the edges by summing the degrees at each vertex. This counts every edge TWICE.

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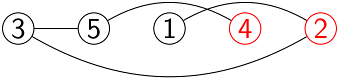
Example:



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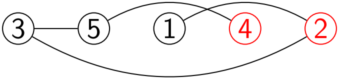

The set of vertices that have NO edges coming out on the right form an Ind Set. Call this set I.

How Big is I?

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WRONG QUESTION!

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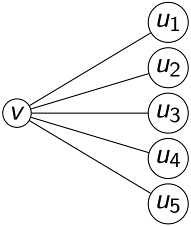
What is the EXPECTED VALUE of the size of I.

(NOTE- we permuted the vertices RANDOMLY)

What is Prob $v \in I$

Let $v \in V$. What is prob that $v \in I$



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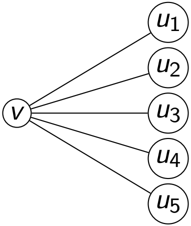

v has degree d_v. How many ways can v and its vertices be laid out: $(d_v + 1)!$. In how many of them is v on the right? $d_v!$.

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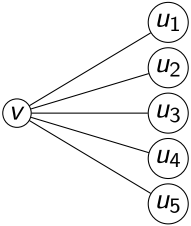

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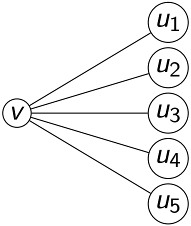

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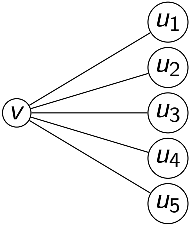

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Hence

$$E(|I|) = \sum_{v \in V} \frac{1}{d_v + 1}.$$

How Big is this Sum?

Need to find lower bound on

$$\sum_{v \in V} \frac{1}{d_v + 1}.$$

Rephrase

NEW PROBLEM:
Minimize

$$\sum_{v \in V} \frac{1}{x_v + 1}$$

relative to the constraint:

$$\sum_{v \in V} x_v = 2e.$$

KNOWN: This sum is minimized when all of the x_v are $\frac{2e}{|V|} = \frac{2e}{n}$.
So the min the sum can be is

$$\sum_{v \in V} \frac{1}{\frac{2e}{n} + 1} = \frac{n}{\frac{2e}{n} + 1}.$$

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$$E(I) \geq \sum_{v \in V} \frac{1}{x_v + 1} \geq \sum_{v \in V} \frac{1}{\frac{2e}{n} + 1} = \frac{n}{\frac{2e}{n} + 1}.$$