Application! Restricting Domains To Stop Being Onto

Exposition by William Gasarch

December 14, 2024

Credit Where Credit Was Due

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He is interested in the logical strength of **The Thin Set Theorem**. This will not be our concern.

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$f : \mathbb{Z} \to \mathbb{Z}$ via $f(x) = x + 1$ is onto
$f : \mathbb{N} \to \mathbb{Z}$ via $f(x) = x + 1$ is NOT onto.

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Good Question For which $f : \mathbb{Z} \rightarrow \mathbb{Z}$ is there an INFINITE set $\mathbb{D} \subseteq \mathbb{Z}$ such that $f : \mathbb{D} \rightarrow \mathbb{Z}$ is not onto.

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That wasn't stupid, but it was easy.

Domain $\mathbb{Z} \times \mathbb{Z}$

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Answer on next page.

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Why 3? We will discuss that later.

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If $\mathrm{COL}_3(H_3) = \mathbf{R}$ then $(\forall x, y \in H_1, x > y)[f(x,y) \notin \{0,1,2\}]$.
We show that f on $H_3 \times H_3$ is not onto.

f Restricted to $H_3 \times H_3$ is Not Onto

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$f(x,y)$ with $x < y$ is either always 0, always 1, always 2, or always $\notin \{0,1,2\}$.

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One of the four colors is not here.

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One of the four colors is not here.

if 0 is not a color of homog set then 0 not in the image, so NOT onto.

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One of the four colors is not here.

if 0 is not a color of homog set then 0 not in the image, so NOT onto.

if 1 is not a color of homog set then 1 not in the image, so NOT onto.

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One of the four colors is not here.

if 0 is not a color of homog set then 0 not in the image, so NOT onto.

if 1 is not a color of homog set then 1 not in the image, so NOT onto.

if 2 is not a color of homog set then 2 not in the image, so NOT onto.

if **R** is not a color of homog set then image is subset of $\{0,1,2\}$, so NOT onto.

f Why [4]?

f Why [4]?

Why four colors?

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Why four colors?

f can have three kinds of input:

f Why $[4]$?

Why four colors?

f can have three kinds of input:

x, x

f **Why** $[4]$**?**

Why four colors?

f can have three kinds of input:

x, x

$x < y$

f Why [4]?

Why four colors?

f can have three kinds of input:

x, x

$x < y$

$y < x$

f Why $[4]$?

Why four colors?

f can have three kinds of input:

x, x

$x < y$

$y < x$

We pick 4 since it is one more than the number possible types of inputs.

f Why $[4]$?

Why four colors?

f can have three kinds of input:

x, x

$x < y$

$y < x$

We pick 4 since it is one more than the number possible types of inputs.

We will discuss this more after we do the 3-hypergraph Ramsey Theorem and can examine $f(x, y, z)$.