

BILL, RECORD LECTURE!!!!

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Euclidean Ramsey Theory: Triangles

Exposition by **William Gasarch**

January 26, 2025

Credit Where Credit is Due

The the main thm of these slides is due to Paul Erdős, Ronald Graham, Peter Montgomery, Bruce L. Rothchild, Joel Spencer, Ernst G. Straus.

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Euclidean Ramsey Theorems I

Journal of Combinatorial Theory (A), Vol. 14, 341-363, 1973

Here is a link.

<https://www.cs.umd.edu/~gasarch/TOPICS/eramsey/eramseyOne.pdf>

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- Answer on next slide

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New Question either a mono $1 - 1 - 1$ **or** mono $2 - 2 - 2$ **or** \dots .

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We prove this rather than $T_1 - T_{\sqrt{3}} - T_2$ since this makes the figures easier to draw.

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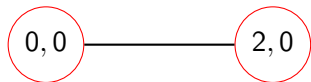
Assume by way of contradiction that there is a $\text{COL}: \mathbb{R}^2 \rightarrow [2]$
with no mono T_2 , $T_{2\sqrt{3}}$ or T_4 .

There are Two **R** Points Two Apart

By Thm from last lecture \exists two points, an inch apart, same color.
We can assume that $(0,0)$ and $(2,0)$ are **R**.

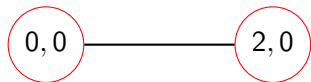
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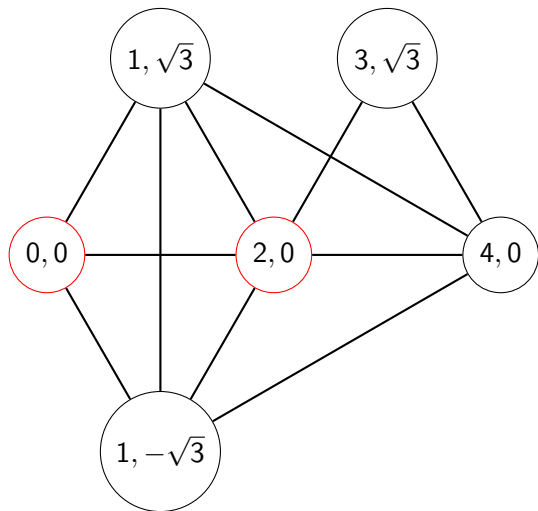
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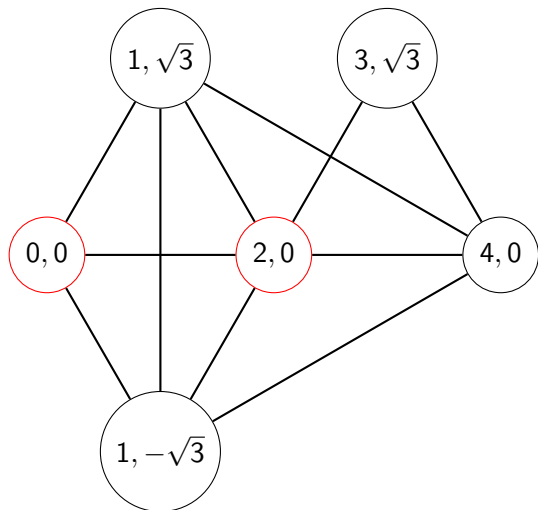
On the next slide we add four more points of interest.

Six Point of Interest

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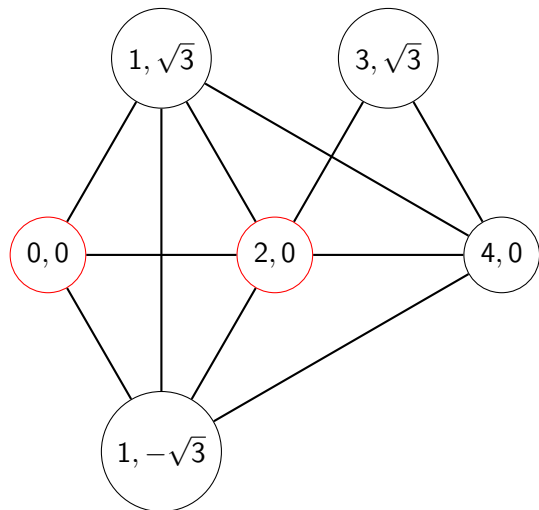


Six Point of Interest



$(0,0) - (1, \sqrt{3}) - (2,0)$ is a T_2 so $\text{COL}(1, \sqrt{3}) = \mathbf{B}$.

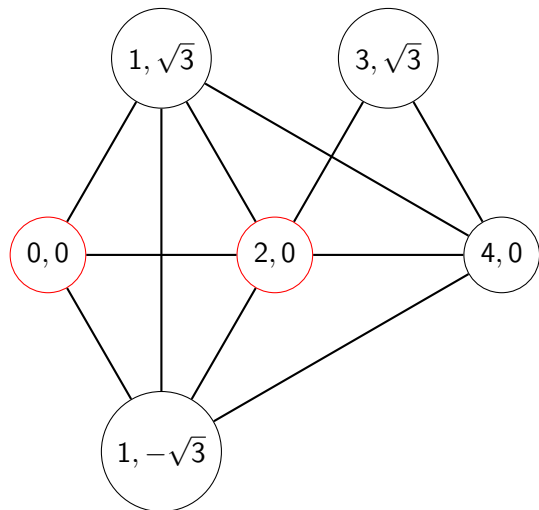
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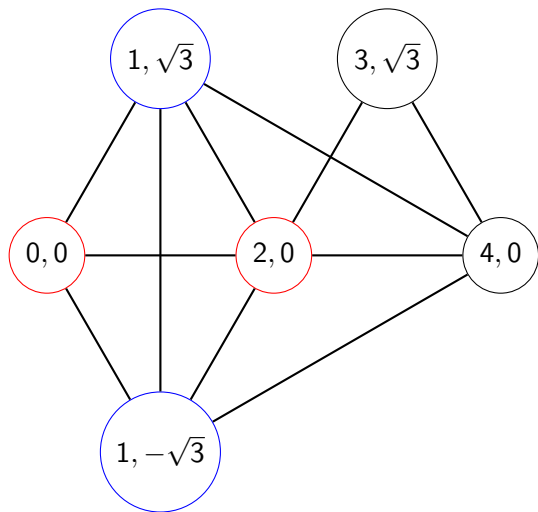


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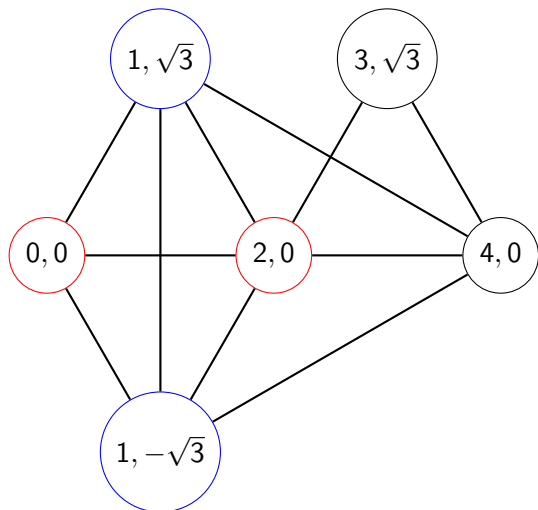
$(0,0) - (1, -\sqrt{3}) - (2,0)$ is a T_2 so $\text{COL}(1, -\sqrt{3}) = \mathbf{B}$.

Next picture has this information.

$(1, \sqrt{3})$ and $(1, -\sqrt{3})$ are **B**

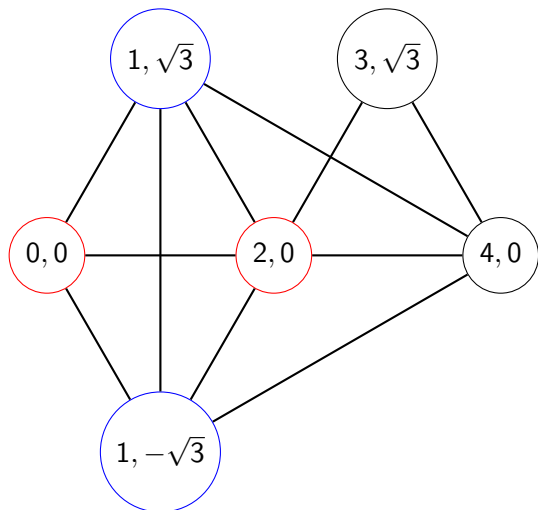


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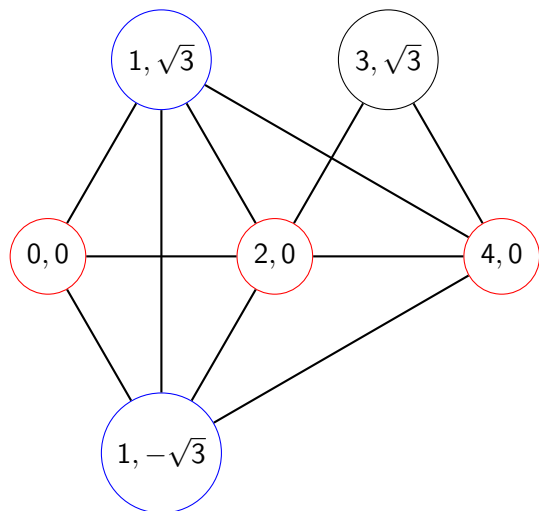
$(1, \sqrt{3}) - (1, -\sqrt{3}) - (4,0)$ is a $T_{2\sqrt{3}}$ so $\text{COL}(4,0) = \mathbf{R}$.

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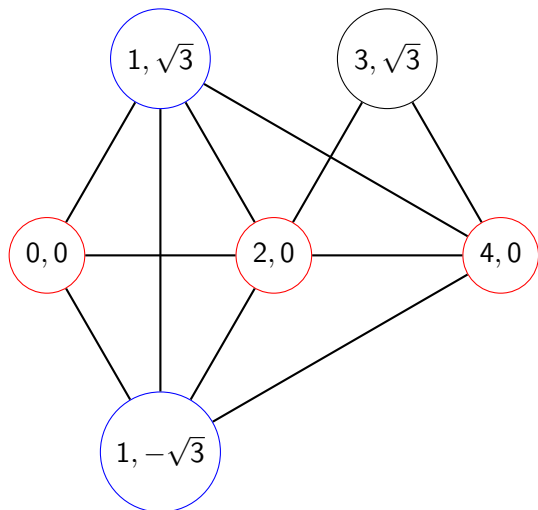


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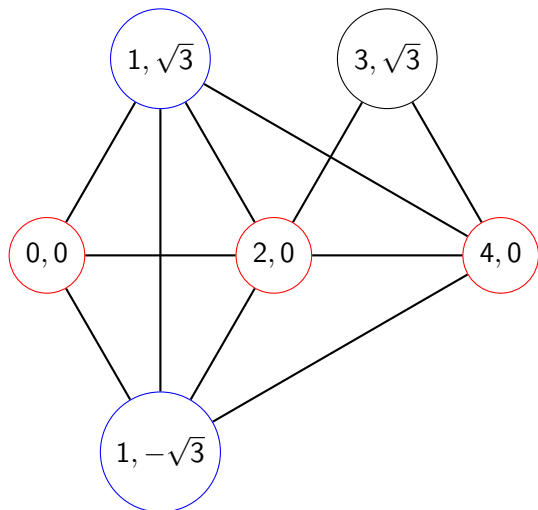


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$(2, 0) - (4, 0) - (3, \sqrt{3})$ is a T_2 so $\text{COL}(3, \sqrt{3}) = \mathbf{B}$.

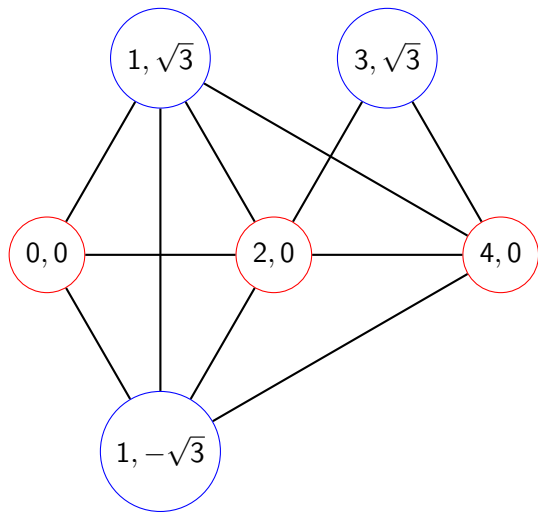
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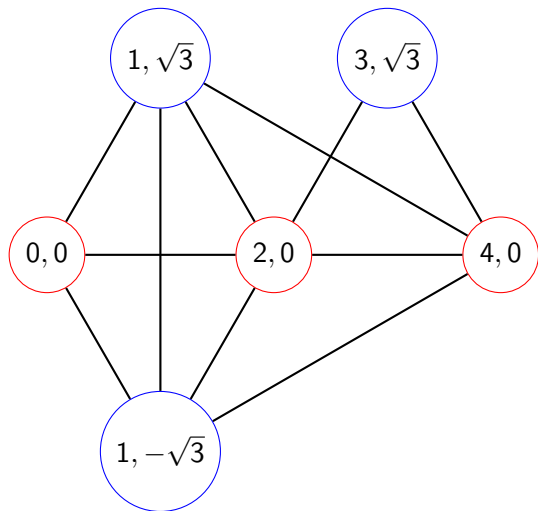
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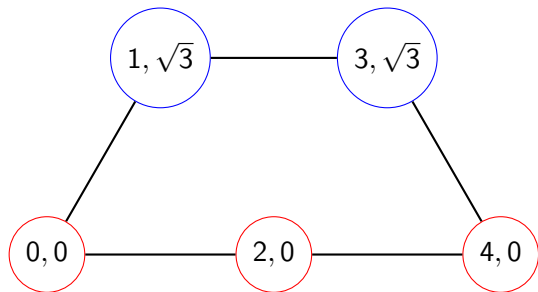


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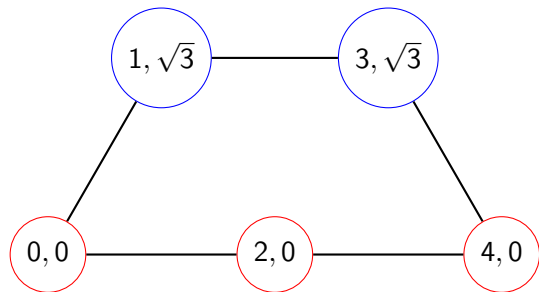


Next picture removes stuff we don't need anymore.

Where We Are Now

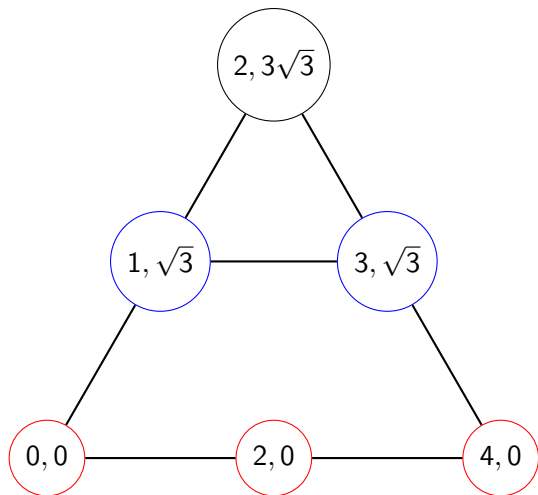


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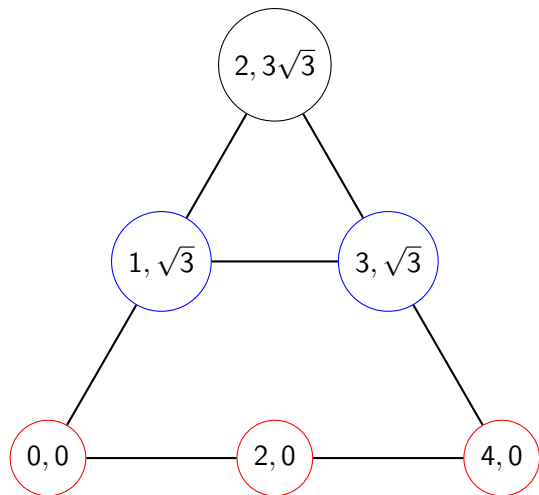


We add a the point $(2, 2\sqrt{3})$ on the next slide.

A Point That Can't be R or B

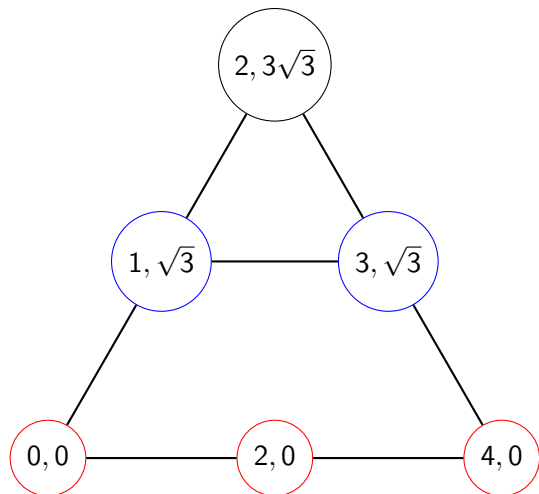


A Point That Can't be **R** or **B**



$(2, 3\sqrt{3}) - (1, \sqrt{3}) - (3, \sqrt{3})$ is a T_2 so $\text{COL}(2, 3\sqrt{3}) \neq \mathbf{B}$.

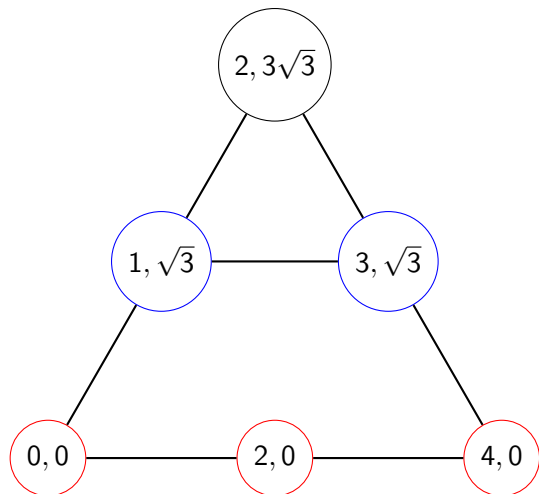
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$(0, 0) - (4, 0) - (2, 3\sqrt{3})$ is a T_4 so $\text{COL}(2, 3\sqrt{3}) \neq \mathbf{R}$.

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$(2, 3\sqrt{3}) - (1, \sqrt{3}) - (3, \sqrt{3})$ is a T_2 so $\text{COL}(2, 3\sqrt{3}) \neq \mathbf{B}$.

$(0, 0) - (4, 0) - (2, 3\sqrt{3})$ is a T_4 so $\text{COL}(2, 3\sqrt{3}) \neq \mathbf{R}$.

$\text{COL}(2, 3\sqrt{3}) \notin \{\mathbf{R}, \mathbf{B}\}$. Contradiction!