Roth's Theorem
A Dense Enough Set Has a 3-AP

Exposition by William Gasarch and Kelin Zhu

December 20, 2024

The Erdös-Turan Conjecture

The Erdös-Turan Conjecture

Def Let $N \in \mathbb{N}$. Let $A \subseteq [N]$. The density of A is $|A|/N$.

The Erdös-Turan Conjecture

Def Let $N \in \mathbb{N}$. Let $A \subseteq [N]$. The density of A is $|A|/N$.

Szemerédi's Thm For all $\delta > 0$, for all k, there exists $N = N(\delta, k)$ such that the following holds:

The Erdös-Turan Conjecture

Def Let $N \in \mathbb{N}$. Let $A \subseteq [N]$. The density of A is $|A|/N$.

Szemerédi's Thm For all $\delta > 0$, for all k, there exists $N = N(\delta, k)$ such that the following holds:

If $A \subseteq [N]$ and A has density $\geq \delta$ then A has a k-AP.

The Erdös-Turan Conjecture

Def Let $N \in \mathbb{N}$. Let $A \subseteq [N]$. The density of A is $|A|/N$.

Szemerédi's Thm For all $\delta > 0$, for all k, there exists $N = N(\delta, k)$ such that the following holds:

If $A \subseteq [N]$ and A has density $\geq \delta$ then A has a k-AP.

We won't do the (hard) proof. We will do:

The Erdös-Turan Conjecture

Def Let $N \in \mathbb{N}$. Let $A \subseteq [N]$. The density of A is $|A|/N$.

Szemerédi's Thm For all $\delta > 0$, for all k, there exists $N = N(\delta, k)$ such that the following holds:

If $A \subseteq [N]$ and A has density $\geq \delta$ then A has a k-AP.

We won't do the (hard) proof. We will do:

1) Some easy cases, and

The Erdös-Turan Conjecture

Def Let $N \in \mathbb{N}$. Let $A \subseteq [N]$. The density of A is $|A|/N$.

Szemerédi's Thm For all $\delta > 0$, for all k, there exists $N = N(\delta, k)$ such that the following holds:

If $A \subseteq [N]$ and A has density $\geq \delta$ then A has a k-AP.

We won't do the (hard) proof. We will do:

1) Some easy cases, and

2) The $k = 3$ case which involves the Discrete Fourier Transform.

An Easy Case

Thm Let $N \geq 3$. Let $A \subseteq [N]$ of density ≥ 0.67. Then A contains a 3-AP.

An Easy Case

Thm Let $N \geq 3$. Let $A \subseteq [N]$ of density ≥ 0.67. Then A contains a 3-AP.

We can assume $N \equiv 0 \pmod{3}$.

An Easy Case

Thm Let $N \geq 3$. Let $A \subseteq [N]$ of density ≥ 0.67. Then A contains a 3-AP.
We can assume $N \equiv 0 \pmod 3$.
Look at

An Easy Case

Thm Let $N \geq 3$. Let $A \subseteq [N]$ of density ≥ 0.67. Then A contains a 3-AP.

We can assume $N \equiv 0 \pmod 3$.

Look at

$$\{1, 2, 3\}, \{4, 5, 6\}, \ldots, \{N-2, N-1, N\}.$$

An Easy Case

Thm Let $N \geq 3$. Let $A \subseteq [N]$ of density ≥ 0.67. Then A contains a 3-AP.
We can assume $N \equiv 0 \pmod 3$.
Look at

$$\{1,2,3\},\{4,5,6\},\ldots,\{N-2,N-1,N\}.$$

Case 1 $\exists x \equiv 1 \pmod 3$, $\{x, x+1, x+2\} \in A$. A has a 3-AP.

An Easy Case

Thm Let $N \geq 3$. Let $A \subseteq [N]$ of density ≥ 0.67. Then A contains a 3-AP.
We can assume $N \equiv 0 \pmod 3$.
Look at

$$\{1, 2, 3\}, \{4, 5, 6\}, \ldots, \{N-2, N-1, N\}.$$

Case 1 $\exists x \equiv 1 \pmod 3$, $\{x, x+1, x+2\} \in A$. A has a 3-AP.

Case 2 $\forall x \equiv 1 \pmod 3$, $|\{x, x+1, x+2\} \cap A| \leq 2$. Then

An Easy Case

Thm Let $N \geq 3$. Let $A \subseteq [N]$ of density ≥ 0.67. Then A contains a 3-AP.
We can assume $N \equiv 0 \pmod 3$.
Look at

$$\{1,2,3\}, \{4,5,6\}, \ldots, \{N-2, N-1, N\}.$$

Case 1 $\exists x \equiv 1 \pmod 3$, $\{x, x+1, x+2\} \in A$. A has a 3-AP.

Case 2 $\forall x \equiv 1 \pmod 3$, $|\{x, x+1, x+2\} \cap A| \leq 2$. Then
$|A| \leq 2 \times \frac{N}{3} \leq 0.667N < 0.67N$

An Easy Case

Thm Let $N \geq 3$. Let $A \subseteq [N]$ of density ≥ 0.67. Then A contains a 3-AP.
We can assume $N \equiv 0 \pmod 3$.
Look at

$$\{1, 2, 3\}, \{4, 5, 6\}, \ldots, \{N-2, N-1, N\}.$$

Case 1 $\exists x \equiv 1 \pmod 3$, $\{x, x+1, x+2\} \in A$. A has a 3-AP.

Case 2 $\forall x \equiv 1 \pmod 3$, $|\{x, x+1, x+2\} \cap A| \leq 2$. Then
$|A| \leq 2 \times \frac{N}{3} \leq 0.667N < 0.67N$
This contradicts A having density ≥ 0.67.

An Easy Case

Thm Let $N \geq 3$. Let $A \subseteq [N]$ of density ≥ 0.67. Then A contains a 3-AP.
We can assume $N \equiv 0 \pmod 3$.
Look at

$$\{1,2,3\},\{4,5,6\},\ldots,\{N-2,N-1,N\}.$$

Case 1 $\exists x \equiv 1 \pmod 3$, $\{x, x+1, x+2\} \in A$. A has a 3-AP.

Case 2 $\forall x \equiv 1 \pmod 3$, $|\{x, x+1, x+2\} \cap A| \leq 2$. Then
$|A| \leq 2 \times \frac{N}{3} \leq 0.667N < 0.67N$
This contradicts A having density ≥ 0.67.

So $N(0.67, 3) = 3$.

An Easy Case

Thm Let $N \geq 3$. Let $A \subseteq [N]$ of density ≥ 0.67. Then A contains a 3-AP.
We can assume $N \equiv 0 \pmod 3$.
Look at

$$\{1,2,3\},\{4,5,6\},\ldots,\{N-2,N-1,N\}.$$

Case 1 $\exists x \equiv 1 \pmod 3$, $\{x,x+1,x+2\} \in A$. A has a 3-AP.

Case 2 $\forall x \equiv 1 \pmod 3$, $|\{x,x+1,x+2\} \cap A| \leq 2$. Then
$|A| \leq 2 \times \frac{N}{3} \leq 0.667N < 0.67N$
This contradicts A having density ≥ 0.67.

So $N(0.67,3) = 3$.

There may be a HW where you are asked to derive bounds on other $N(\delta,k)$.

Upper Bounds on VDW Numbers: 1974-1988

Upper Bounds on VDW Numbers: 1974-1988

4) 1974: Szemerédi proves ET-conj for $k = 4$.

Upper Bounds on VDW Numbers: 1974-1988

4) 1974: Szemerédi proves ET-conj for $k = 4$.
Proof is combinatorial but difficult.

Upper Bounds on VDW Numbers: 1974-1988

4) 1974: Szemerédi proves ET-conj for $k = 4$.
Proof is combinatorial but difficult.
Uses VDW theorem so does not give better bounds on $W(4, c)$.

Upper Bounds on VDW Numbers: 1974-1988

4) 1974: Szemerédi proves ET-conj for $k = 4$.
Proof is combinatorial but difficult.
Uses VDW theorem so does not give better bounds on $W(4, c)$.

5) 1975: Szemerédi proves ET-conj.

Upper Bounds on VDW Numbers: 1974-1988

4) 1974: Szemerédi proves ET-conj for $k = 4$.
Proof is combinatorial but difficult.
Uses VDW theorem so does not give better bounds on $W(4, c)$.

5) 1975: Szemerédi proves ET-conj.

Proof is combinatorial but difficult.

Upper Bounds on VDW Numbers: 1974-1988

4) 1974: Szemerédi proves ET-conj for $k = 4$.
Proof is combinatorial but difficult.
Uses VDW theorem so does not give better bounds on $W(4, c)$.

5) 1975: Szemerédi proves ET-conj.

Proof is combinatorial but difficult.

Uses VDW's theorem so does not give better bounds on $W(k, c)$.

Upper Bounds on VDW Numbers: 1974-1988

4) 1974: Szemerédi proves ET-conj for $k = 4$.
Proof is combinatorial but difficult.
Uses VDW theorem so does not give better bounds on $W(4, c)$.

5) 1975: Szemerédi proves ET-conj.

Proof is combinatorial but difficult.

Uses VDW's theorem so does not give better bounds on $W(k, c)$.

The ET-Conj is now called Szemerédi's Theorem.

Upper Bounds on VDW Numbers: 1974-1988

4) 1974: Szemerédi proves ET-conj for $k = 4$.
Proof is combinatorial but difficult.
Uses VDW theorem so does not give better bounds on $W(4, c)$.

5) 1975: Szemerédi proves ET-conj.

Proof is combinatorial but difficult.

Uses VDW's theorem so does not give better bounds on $W(k, c)$.

The ET-Conj is now called Szemerédi's Theorem.

6) 1977: Furstenberg proves ET-conj using Ergodic methods.

Upper Bounds on VDW Numbers: 1974-1988

4) 1974: Szemerédi proves ET-conj for $k = 4$.
Proof is combinatorial but difficult.
Uses VDW theorem so does not give better bounds on $W(4, c)$.

5) 1975: Szemerédi proves ET-conj.

Proof is combinatorial but difficult.

Uses VDW's theorem so does not give better bounds on $W(k, c)$.

The ET-Conj is now called Szemerédi's Theorem.

6) 1977: Furstenberg proves ET-conj using Ergodic methods.

Proof does not gives bounds on $W(k, c)$.

Upper Bounds on VDW Numbers: 1974-1988

4) 1974: Szemerédi proves ET-conj for $k = 4$.
Proof is combinatorial but difficult.
Uses VDW theorem so does not give better bounds on $W(4, c)$.

5) 1975: Szemerédi proves ET-conj.

Proof is combinatorial but difficult.

Uses VDW's theorem so does not give better bounds on $W(k, c)$.

The ET-Conj is now called Szemerédi's Theorem.

6) 1977: Furstenberg proves ET-conj using Ergodic methods.

Proof does not gives bounds on $W(k, c)$.

Proof theorists extract bounds from proof which are worse than VDW's bounds.

Upper Bounds on VDW Numbers: 1974-1988

4) 1974: Szemerédi proves ET-conj for $k = 4$.
Proof is combinatorial but difficult.
Uses VDW theorem so does not give better bounds on $W(4, c)$.

5) 1975: Szemerédi proves ET-conj.

Proof is combinatorial but difficult.

Uses VDW's theorem so does not give better bounds on $W(k, c)$.

The ET-Conj is now called Szemerédi's Theorem.

6) 1977: Furstenberg proves ET-conj using Ergodic methods.

Proof does not gives bounds on $W(k, c)$.

Proof theorists extract bounds from proof which are worse than VDW's bounds.

7) 1988: Shelah new proof of VDW which gives prim rec bounds.

Upper Bounds on VDW Numbers: 1974-1988

4) 1974: Szemerédi proves ET-conj for $k = 4$.
Proof is combinatorial but difficult.
Uses VDW theorem so does not give better bounds on $W(4, c)$.

5) 1975: Szemerédi proves ET-conj.

Proof is combinatorial but difficult.

Uses VDW's theorem so does not give better bounds on $W(k, c)$.

The ET-Conj is now called Szemerédi's Theorem.

6) 1977: Furstenberg proves ET-conj using Ergodic methods.

Proof does not gives bounds on $W(k, c)$.

Proof theorists extract bounds from proof which are worse than VDW's bounds.

7) 1988: Shelah new proof of VDW which gives prim rec bounds.

Proof is elementary and does not use any of the ET stuff.

Upper Bounds on VDW Numbers: 1974-1988

4) 1974: Szemerédi proves ET-conj for $k = 4$.
Proof is combinatorial but difficult.
Uses VDW theorem so does not give better bounds on $W(4, c)$.

5) 1975: Szemerédi proves ET-conj.

Proof is combinatorial but difficult.

Uses VDW's theorem so does not give better bounds on $W(k, c)$.

The ET-Conj is now called Szemerédi's Theorem.

6) 1977: Furstenberg proves ET-conj using Ergodic methods.

Proof does not gives bounds on $W(k, c)$.

Proof theorists extract bounds from proof which are worse than VDW's bounds.

7) 1988: Shelah new proof of VDW which gives prim rec bounds.

Proof is elementary and does not use any of the ET stuff.

He proved Hales-Jewitt Thm which implies VDW's Thm.

Upper Bounds on VDW Numbers: 1974-1988

Upper Bounds on VDW Numbers: 1974-1988

8) 2001: Gowers proves Szemerédi's Thm a new way using combinatorics and Fourier Analysis to obtain the following:

Upper Bounds on VDW Numbers: 1974-1988

8) 2001: Gowers proves Szemerédi's Thm a new way using combinatorics and Fourier Analysis to obtain the following:

$$W(k,c) \le 2^{2^{c^{2^{2^{k+9}}}}}.$$