Fourier Transform and Roth's Theorem

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Outline

Background

Theorem (VDW)

For all positive integers r and k, there exists positive integer N such that any r-coloring of $[N] = \{0, \ldots, N-1\}$ contains a monochromatic k-AP.

Can we go further?

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ET would imply a mono k-AP of each color in VDW.

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Today: new upper bounds on $r_k(N)$ established in 2024 using high-powered analysis

Discrete Fourier Transform

Definition (DFT)

Let Z_N denote the integers modulo N. Also let $\chi(z) = e^{\frac{-2\pi iz}{N}}$. Then, the DFT of a function $f : Z_N \to \mathbb{C}$, denoted as $\widehat{f}$, is defined as:

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We will work in Z_N. Although not all Z_N APs are $\mathbb{Z}$ APs, we will place restrictions on the Z_N AP we find so that it is also a $\mathbb{Z}$ AP.

Theorem (Plancherel)

$$\sum_{x \in Z_N} |f(x)|^2 = \frac{1}{N} \sum_{m \in Z_N} |\widehat{f}(m)|^2$$

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Theorem (Convolution (unconventional))

Define

$$(f * g)(x) = \sum_{y \in Z_N} f(y)g(x - y)$$

Then for any m,

$$\widehat{(f * g)}(m) = \widehat{f}(m)\widehat{g}(m)$$

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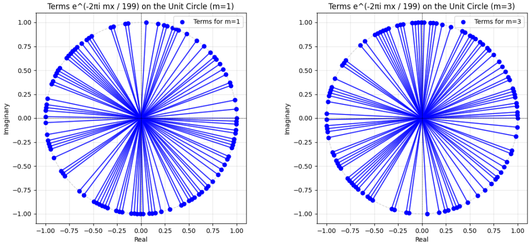

Figure: Two fourier coefficients of the quadratic residues mod 199 - an example of a set with small fourier coefficients

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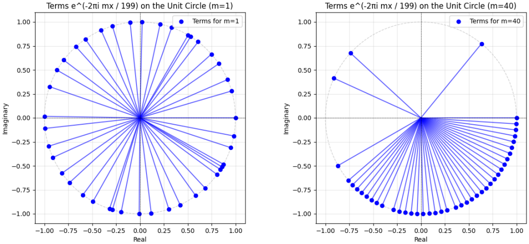

Figure: Example of a large fourier coefficient

Hence, we can center an AP of "length" cN at the "occupied arc" so that it has a high density intersection with mA. We will prove that we can choose the common difference so that the AP consists of enough terms and doesn't "roll over" (isn't split into multiple $\mathbb{Z}$ APs) when we undo multiplication by m.

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If N is large enough that we can repeat this process infinitely, A will have infinite density in some AP, impossible.

Initial Setup

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$$Q = \sum_{x,y,z,x+y\equiv 2z} B(x)B(y)A(z)$$

$$= \frac{1}{N} \sum_{x,y,z,m} B(x)B(y)A(z)\chi(-m(x + z - 2y))$$

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$$\begin{aligned} Q &= \sum_{x,y,z,x+y\equiv 2z} B(x)B(y)A(z) \\ &= \frac{1}{N}\sum_{x,y,z,m} B(x)B(y)A(z)\chi(-m(x+z-2y)) \\ &= \frac{1}{N}\sum_{m} \widehat{B}(m)\widehat{B}(-2m)\widehat{A}(m) \end{aligned}$$

At this point, we split the sum into $\frac{1}{N}|B|^2|A| + \sum_{m\neq 0} \widehat{B}(m)\widehat{B}(-2m)\widehat{A}(m)$. The term $\frac{1}{N}\sum_{m\neq 0} \widehat{B}(m)\widehat{B}(-2m)\widehat{A}(m)$ will be denoted by the "error term" E.

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Apply Cauchy-Schwartz and Plancherel:

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\begin{aligned}
|E| &\le \frac{1}{N} \max_{m\neq 0} |\widehat{A}(m)| \left| \sum_{m\neq 0} \widehat{B}(m)\widehat{B}(-2m) \right| \\
&\le \frac{1}{N} \max_{m\neq 0} |\widehat{A}(m)| \sum_{m} |\widehat{B}(m)\widehat{B}(-2m)| \\
&\le \frac{1}{N} \max_{m\neq 0} |\widehat{A}(m)| \left(\sum_{m} |\widehat{B}(m)|^2 \right)^{\frac{1}{2}} \left(\sum_{m} |\widehat{B}(-2m)|^2 \right)^{\frac{1}{2}} \\
&= \max_{m\neq 0} |\widehat{A}(m)| \left(\sum_{m} |B(m)|^2 \right)^{\frac{1}{2}} \left(\sum_{m} |B(-2m)|^2 \right)^{\frac{1}{2}} \\
&= \max_{m\neq 0} |\widehat{A}(m)| |B|
\end{aligned}
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If we suppose that $\max_{m \neq 0} |\widehat{A}(m)| \leq \frac{\delta^2}{10} N$ and $|B| \geq \frac{|A|}{5}$. we obtain that $|E| \leq \frac{1}{2N} |B|^2 |A|$, and consequentially

$$Q = \frac{1}{N}|B|^2|A| + E \geq \frac{1}{N}|B|^2|A| - |E| \geq \frac{\delta^3}{50} N^2$$

Hence, there are at least $\frac{\delta^3}{50} N^2 - \delta N$ 3-APs in A (accounting for the overcounted $x = y = z$), which is positive as long as $N > \frac{50}{\delta^2}$.

Large Fourier Coefficients

Suppose $|B| < \frac{|A|}{5}$, then if we assume WLOG that $|A \cap [0, \frac{N}{3})| \geq A \cap [\frac{2N}{3}, N)|$, we have $\frac{|A \cap [0, \frac{N}{3})|}{|[0, \frac{N}{3})|} \geq \delta + \frac{\delta}{5}$.

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Now, suppose $\max_{m \neq 0} |\widehat{A}(m)| > \frac{\delta^2}{10} N$. Let the maximum-attaining value of m be r. Pigeonhole: there exist $0 \leq p < q \leq \sqrt{N}$ such that $p - q \leq \sqrt{N}$ and $r(p-q) \leq \sqrt{N} \pmod{N}$.

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Let $d = p - q$. Then, consider the AP between $-\left\lfloor \frac{\sqrt{N}}{6} \right\rfloor d$ and $\left\lfloor \frac{\sqrt{N}}{6} \right\rfloor d$ with common difference d; let the set of its members be P.

$$
\begin{aligned}
|\widehat{P}(r)| &= \left|\sum_{x\in P} \chi(-rx)\right| \\
&\geq Re\left(\sum_{x\in P} \chi(-rx)\right) \\
&\geq \frac{|P|}{2}
\end{aligned}
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Because $|-rx| \leq \frac{N}{6}$, so $Re(e^{\frac{-2\pi irx}{N}}) \geq Re(e^{\frac{\mp 2\pi i}{6}}) = \frac{1}{2}$.

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Let $f(x) = A(x) - \delta$, and $g(x) = f * P(x)$. Note that $\widehat{f}(r) = \widehat{A}(r)$, and

$$\sum_m |g(m)| \geq |\widehat{g}(r)| = |\widehat{f}(r)||\widehat{P}(r)| \geq \frac{\delta^2}{20}N|P|$$

Since f has mean value zero, g also does (expand the sum), so

$$\max g(m) \geq \frac{\sum_{m,g(m)\geq 0} g(m)}{N} = \frac{\sum_m |g(m)|}{2N} = \frac{\delta^2}{40}|P|$$

Let the maximum-attaining m be x. By definition,

$$\begin{aligned}\frac{\delta^2}{40}|P| &\leq \sum_y f(y)P(x-y) \\ &= \sum_y A(y)P(x-y) - \sum_y \delta P(x-y) \\ &= |A \cap (x-P)| - \delta|x-P|\end{aligned}$$

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We've shown that the density of A in $x - P$ is at least $\delta + \frac{\delta^2}{40}$. Also, note that the "difference" between the first and last terms of P, and therefore also of $x - P$, is $2\left\lfloor \frac{\sqrt{N}}{6} \right\rfloor d < N$, so $x - P$ "rolls over" from $n - 1$ to 0 at most once in $\mathbb{Z}$.

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We can write $x - P = P_1 \cup P_2$ where P_1, P_2 are APs in $\mathbb{N}$. Easy to show that one of them (P') has size at least $\frac{\delta^2}{80}|P|$ and satisfies $|A \cap P'| \geq \left(\delta + \frac{\delta^2}{80}\right)|P'|$

Finish

We've shown:

Proposition

As long as $N > \frac{50}{\delta^2}$, we can either find a 3-AP in A ($\star$) or find an AP P' such that $|P'| = \Omega(\sqrt{N})$ and $|A \cap P'| \geq \left(\delta + \frac{\delta^2}{80}\right)|P'|$

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We can replace $[N]$ with P' and reiterate - if N is large enough and we never reach $\star$, we get some AP P''''''' such that A has density > 1 in P''''''', which is impossible.

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Handout - $N = \exp(\exp(c\delta^{-1}))$ suffices.