When Does a 2-Coloring Yield a Mono Unit Square?

Exposition by William Gasarch

November 20, 2024

Credit Where Credit is Due!

The main theorem of these slides is due to **Stefan Burr.**

Credit Where Credit is Due!

The main theorem of these slides is due to **Stefan Burr.**

He did not publish it. It appeared, and credited to him, in

Credit Where Credit is Due!

The main theorem of these slides is due to **Stefan Burr.**

He did not publish it. It appeared, and credited to him, in

Euclidean Ramsey Theorems I

Credit Where Credit is Due!

The main theorem of these slides is due to **Stefan Burr.**

He did not publish it. It appeared, and credited to him, in

Euclidean Ramsey Theorems I

by

Paul Erdös, Ronald Graham, Peter Montgomery, Bruce L. Rothchild, Joel Spencer, Ernst G. Straus.

Credit Where Credit is Due!

The main theorem of these slides is due to **Stefan Burr.**

He did not publish it. It appeared, and credited to him, in

Euclidean Ramsey Theorems I

by
Paul Erdös, Ronald Graham, Peter Montgomery, Bruce L. Rothchild, Joel Spencer, Ernst G. Straus.

Journal of Combinatorial Theory (A), Vol. 14, 341-363, 1973

`https://www.cs.umd.edu/~gasarch/TOPICS/eramsey/eramseyOne.pdf`

What a Mono Unit Square?

Def a **Mono Unit Square** is a unit square with all four corners the same color.

What a Mono Unit Square?

Def a **Mono Unit Square** is a unit square with all four corners the same color.

Def A coloring is **proper** if there is no unit square.

What a Mono Unit Square?

Def a **Mono Unit Square** is a unit square with all four corners the same color.

Def A coloring is **proper** if there is no unit square.

Question Is there a proper 2-coloring of $\mathbb{R}^2$?

What a Mono Unit Square?

Def a **Mono Unit Square** is a unit square with all four corners the same color.

Def A coloring is **proper** if there is no unit square.

Question Is there a proper 2-coloring of $\mathbb{R}^2$?

Answer Yes. We leave this for an exercise.

What About Higher Dimensions?

What About Higher Dimensions?

Vote

What About Higher Dimensions?

Vote

1) There is a proper 2-col of $\mathbb{R}^2$ but not $\mathbb{R}^3$.

What About Higher Dimensions?

Vote

1) There is a proper 2-col of $\mathbb{R}^2$ but not $\mathbb{R}^3$.
2) There is a proper 2-col of $\mathbb{R}^3$ but not $\mathbb{R}^4$.

What About Higher Dimensions?

Vote

1) There is a proper 2-col of $\mathbb{R}^2$ but not $\mathbb{R}^3$.
2) There is a proper 2-col of $\mathbb{R}^3$ but not $\mathbb{R}^4$.
3) There is a proper 2-col of $\mathbb{R}^4$ but not $\mathbb{R}^5$.

What About Higher Dimensions?

Vote

1) There is a proper 2-col of $\mathbb{R}^2$ but not $\mathbb{R}^3$.
2) There is a proper 2-col of $\mathbb{R}^3$ but not $\mathbb{R}^4$.
3) There is a proper 2-col of $\mathbb{R}^4$ but not $\mathbb{R}^5$.
4) There is a proper 2-col of $\mathbb{R}^5$ but not $\mathbb{R}^6$.

What About Higher Dimensions?

Vote

1) There is a proper 2-col of $\mathbb{R}^2$ but not $\mathbb{R}^3$.
2) There is a proper 2-col of $\mathbb{R}^3$ but not $\mathbb{R}^4$.
3) There is a proper 2-col of $\mathbb{R}^4$ but not $\mathbb{R}^5$.
4) There is a proper 2-col of $\mathbb{R}^5$ but not $\mathbb{R}^6$.
5) The exact cutoff is Unknown to Science!

What About Higher Dimensions?

Vote

1) There is a proper 2-col of $\mathbb{R}^2$ but not $\mathbb{R}^3$.
2) There is a proper 2-col of $\mathbb{R}^3$ but not $\mathbb{R}^4$.
3) There is a proper 2-col of $\mathbb{R}^4$ but not $\mathbb{R}^5$.
4) There is a proper 2-col of $\mathbb{R}^5$ but not $\mathbb{R}^6$.
5) The exact cutoff is Unknown to Science!

The answer is on the next slide.

The Exact Cutoff is Unknown To Science!

The Exact Cutoff is Unknown To Science!

Here is all that is known:

The Exact Cutoff is Unknown To Science!

Here is all that is known:

- There is a proper 2-col of $\mathbb{R}^2$.

The Exact Cutoff is Unknown To Science!

Here is all that is known:

- There is a proper 2-col of $\mathbb{R}^2$.
- There is no proper 2-col of $\mathbb{R}^4$.

The Exact Cutoff is Unknown To Science!

Here is all that is known:

- There is a proper 2-col of $\mathbb{R}^2$.
- There is no proper 2-col of $\mathbb{R}^4$.

The proof is a bit beyond this class so we prove the following instead: We will show that

The Exact Cutoff is Unknown To Science!

Here is all that is known:

- There is a proper 2-col of $\mathbb{R}^2$.
- There is no proper 2-col of $\mathbb{R}^4$.

The proof is a bit beyond this class so we prove the following instead: We will show that
For all $\mathrm{COL}\colon \mathbb{R}^6 \rightarrow [2]$ there exists a Mono Unit Square.

The Exact Cutoff is Unknown To Science!

Here is all that is known:

- There is a proper 2-col of $\mathbb{R}^2$.
- There is no proper 2-col of $\mathbb{R}^4$.

The proof is a bit beyond this class so we prove the following instead: We will show that
For all $\mathrm{COL}\colon \mathbb{R}^6 \rightarrow [2]$ there exists a Mono Unit Square.
For all $\mathrm{COL}\colon \mathbb{R}^5 \rightarrow [2]$ there exists a Mono Unit Square.

The Exact Cutoff is Unknown To Science!

Here is all that is known:

- There is a proper 2-col of $\mathbb{R}^2$.
- There is no proper 2-col of $\mathbb{R}^4$.

The proof is a bit beyond this class so we prove the following instead: We will show that
For all $\mathrm{COL}\colon \mathbb{R}^6 \rightarrow [2]$ there exists a Mono Unit Square.
For all $\mathrm{COL}\colon \mathbb{R}^5 \rightarrow [2]$ there exists a Mono Unit Square.
The $\mathbb{R}^5$ result is really an observation about the $\mathbb{R}^6$ proof.

The Exact Cutoff is Unknown To Science!

Here is all that is known:

- There is a proper 2-col of $\mathbb{R}^2$.
- There is no proper 2-col of $\mathbb{R}^4$.

The proof is a bit beyond this class so we prove the following instead: We will show that
For all $\mathrm{COL}\colon \mathbb{R}^6 \rightarrow [2]$ there exists a Mono Unit Square.
For all $\mathrm{COL}\colon \mathbb{R}^5 \rightarrow [2]$ there exists a Mono Unit Square.
The $\mathbb{R}^5$ result is really an observation about the $\mathbb{R}^6$ proof.
We will also have comments on the $\mathbb{R}^4$ proof.

The Exact Cutoff is Unknown To Science!

The following theorem is due to Stefan Burr, as noted earlier.

The Exact Cutoff is Unknown To Science!

The following theorem is due to Stefan Burr, as noted earlier.

Thm For all $\mathrm{COL}\colon \mathbb{R}^6 \rightarrow [2]$ there exists a Mono Unit Square.

The Exact Cutoff is Unknown To Science!

The following theorem is due to Stefan Burr, as noted earlier.
Thm For all $\mathrm{COL}\colon \mathbb{R}^6 \rightarrow [2]$ there exists a Mono Unit Square.

Let $\mathrm{COL}\colon \mathbb{R}^6 \rightarrow [2]$.

The Exact Cutoff is Unknown To Science!

The following theorem is due to Stefan Burr, as noted earlier.
Thm For all $\mathrm{COL}\colon \mathbb{R}^6 \to [2]$ there exists a Mono Unit Square.

Let $\mathrm{COL}\colon \mathbb{R}^6 \to [2]$.

We form a coloring $\mathrm{COL}'\colon \binom{[6]}{2} \to [2]$.

We Look at 15 Points in $\mathbb{R}^6$

We look at the following 15 points of $\mathbb{R}^6$.

We Look at 15 Points in $\mathbb{R}^6$

We look at the following 15 points of $\mathbb{R}^6$.
$p_{1,2} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0, 0, 0)$.

We Look at 15 Points in $\mathbb{R}^6$

We look at the following 15 points of $\mathbb{R}^6$.
$p_{1,2} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0, 0, 0)$.
$p_{1,3} = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0, 0, 0)$.

We Look at 15 Points in $\mathbb{R}^6$

We look at the following 15 points of $\mathbb{R}^6$.

$p_{1,2} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0, 0, 0)$.

$p_{1,3} = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0, 0, 0)$.

$\vdots$

We Look at 15 Points in $\mathbb{R}^6$

We look at the following 15 points of $\mathbb{R}^6$.

$p_{1,2} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0, 0, 0).$

$p_{1,3} = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0, 0, 0).$

$\vdots$

$p_{5,6} = (0, 0, 0, 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}).$

We Look at 15 Points in $\mathbb{R}^6$

We look at the following 15 points of $\mathbb{R}^6$.

$p_{1,2} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0, 0, 0)$.

$p_{1,3} = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0, 0, 0)$.

$\vdots$

$p_{5,6} = (0, 0, 0, 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

Define $\mathrm{COL}'(i,j) = \mathrm{COL}(p_{i,j})$.

Use Mono C_4 Theorem

C_4 Thm For all 2-colorings of $\binom{[6]}{2} \rightarrow [2]$ there is a mono C_4.

Use Mono C_4 Theorem

C_4 Thm For all 2-colorings of $\binom{[6]}{2} \rightarrow [2]$ there is a mono C_4.

By Thm, COL' has a mono C_4. We assume

Use Mono C_4 Theorem

C_4 Thm For all 2-colorings of $\binom{[6]}{2} \rightarrow [2]$ there is a mono C_4.

By Thm, COL' has a mono C_4. We assume

$\mathrm{COL}'(1,2) = \mathrm{COL}'(2,3) = \mathrm{COL}'(3,4) = \mathrm{COL}'(4,1) =$ **R**

Use Mono C_4 Theorem

C_4 Thm For all 2-colorings of $\binom{[6]}{2} \rightarrow [2]$ there is a mono C_4.

By Thm, COL' has a mono C_4. We assume

$\mathrm{COL}'(1,2) = \mathrm{COL}'(2,3) = \mathrm{COL}'(3,4) = \mathrm{COL}'(4,1) =$ **R**

Hence

$\mathrm{COL}(p_{1,2}) = \mathrm{COL}(p_{2,3}) = \mathrm{COL}(p_{3,4}) = \mathrm{COL}(p_{4,1}) =$ **R**

Use Mono C_4 Theorem

C_4 Thm For all 2-colorings of $\binom{[6]}{2} \rightarrow [2]$ there is a mono C_4.

By Thm, COL' has a mono C_4. We assume

$\mathrm{COL}'(1,2) = \mathrm{COL}'(2,3) = \mathrm{COL}'(3,4) = \mathrm{COL}'(4,1) = \mathbf{R}$

Hence

$\mathrm{COL}(p_{1,2}) = \mathrm{COL}(p_{2,3}) = \mathrm{COL}(p_{3,4}) = \mathrm{COL}(p_{4,1}) = \mathbf{R}$

These points form a unit square:

Use Mono C_4 Theorem

C_4 Thm For all 2-colorings of $\binom{[6]}{2} \rightarrow [2]$ there is a mono C_4.

By Thm, COL' has a mono C_4. We assume

$\mathrm{COL}'(1,2) = \mathrm{COL}'(2,3) = \mathrm{COL}'(3,4) = \mathrm{COL}'(4,1) = \mathbf{R}$

Hence

$\mathrm{COL}(p_{1,2}) = \mathrm{COL}(p_{2,3}) = \mathrm{COL}(p_{3,4}) = \mathrm{COL}(p_{4,1}) = \mathbf{R}$

These points form a unit square:

$p_{i,i+1}$ and $p_{i+1,i+2}$

Use Mono C_4 Theorem

C_4 Thm For all 2-colorings of $\binom{[6]}{2} \rightarrow [2]$ there is a mono C_4.

By Thm, COL' has a mono C_4. We assume

$\mathrm{COL}'(1,2) = \mathrm{COL}'(2,3) = \mathrm{COL}'(3,4) = \mathrm{COL}'(4,1) =$ **R**

Hence

$\mathrm{COL}(p_{1,2}) = \mathrm{COL}(p_{2,3}) = \mathrm{COL}(p_{3,4}) = \mathrm{COL}(p_{4,1}) =$ **R**

These points form a unit square:

$p_{i,i+1}$ and $p_{i+1,i+2}$

On ith coordinate $p_{i,i+1}$ is f, $p_{i+1,i+2}$ is 0.

Use Mono C_4 Theorem

C_4 Thm For all 2-colorings of $\binom{[6]}{2} \rightarrow [2]$ there is a mono C_4.

By Thm, COL' has a mono C_4. We assume

$\mathrm{COL}'(1,2) = \mathrm{COL}'(2,3) = \mathrm{COL}'(3,4) = \mathrm{COL}'(4,1) =$ **R**

Hence

$\mathrm{COL}(p_{1,2}) = \mathrm{COL}(p_{2,3}) = \mathrm{COL}(p_{3,4}) = \mathrm{COL}(p_{4,1}) =$ **R**

These points form a unit square:

$p_{i,i+1}$ and $p_{i+1,i+2}$

On ith coordinate $p_{i,i+1}$ is f, $p_{i+1,i+2}$ is 0.

On ith coordinate $p_{i,i+1}$ is 0, $p_{i+1,i+2}$ is $\frac{1}{\sqrt{2}}$.

Use Mono C_4 Theorem

C_4 Thm For all 2-colorings of $\binom{[6]}{2} \rightarrow [2]$ there is a mono C_4.

By Thm, COL' has a mono C_4. We assume

$\mathrm{COL}'(1,2) = \mathrm{COL}'(2,3) = \mathrm{COL}'(3,4) = \mathrm{COL}'(4,1) =$ **R**

Hence

$\mathrm{COL}(p_{1,2}) = \mathrm{COL}(p_{2,3}) = \mathrm{COL}(p_{3,4}) = \mathrm{COL}(p_{4,1}) =$ **R**

These points form a unit square:

$p_{i,i+1}$ and $p_{i+1,i+2}$

On ith coordinate $p_{i,i+1}$ is f, $p_{i+1,i+2}$ is 0.

On ith coordinate $p_{i,i+1}$ is 0, $p_{i+1,i+2}$ is $\frac{1}{\sqrt{2}}$.

On all other coordinates $p_{i,i+1}$ and $p_{i+1,i+2}$ agree.

Use Mono C_4 Theorem

C_4 Thm For all 2-colorings of $\binom{[6]}{2} \rightarrow [2]$ there is a mono C_4.

By Thm, COL' has a mono C_4. We assume

$\mathrm{COL}'(1,2) = \mathrm{COL}'(2,3) = \mathrm{COL}'(3,4) = \mathrm{COL}'(4,1) =$ **R**

Hence

$\mathrm{COL}(p_{1,2}) = \mathrm{COL}(p_{2,3}) = \mathrm{COL}(p_{3,4}) = \mathrm{COL}(p_{4,1}) =$ **R**

These points form a unit square:

$p_{i,i+1}$ and $p_{i+1,i+2}$

On ith coordinate $p_{i,i+1}$ is f, $p_{i+1,i+2}$ is 0.

On ith coordinate $p_{i,i+1}$ is 0, $p_{i+1,i+2}$ is $\frac{1}{\sqrt{2}}$.

On all other coordinates $p_{i,i+1}$ and $p_{i+1,i+2}$ agree.

Hence $d(p_{i,i+1}, p_{i+1,i+2}) = \sqrt{(\frac{1}{\sqrt{2}})^2 + (\frac{1}{\sqrt{2}})^2} = 1.$

Improvements On $\mathbb{R}^6$

What About $\mathbb{R}^5$?

Observation The 15 vectors

What About $\mathbb{R}^5$?

Observation The 15 vectors

$p_{1,2} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0, 0, 0)$,

What About $\mathbb{R}^5$?

Observation The 15 vectors

$p_{1,2} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0, 0, 0)$, $p_{1,3} = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0, 0, 0)$,

What About $\mathbb{R}^5$?

Observation The 15 vectors
$p_{1,2} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0, 0, 0)$, $p_{1,3} = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0, 0, 0)$, $\cdots$,
$p_{5,6} = (0, 0, 0, 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

What About $\mathbb{R}^5$?

Observation The 15 vectors
$p_{1,2} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0, 0, 0)$, $p_{1,3} = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0, 0, 0)$, $\cdots$,
$p_{5,6} = (0, 0, 0, 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.
Span a 5-dimensional subspace H of $\mathbb{R}^6$.

What About $\mathbb{R}^5$?

Observation The 15 vectors
$p_{1,2} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0, 0, 0)$, $p_{1,3} = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0, 0, 0)$, $\cdots$,
$p_{5,6} = (0, 0, 0, 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.
Span a 5-dimensional subspace H of $\mathbb{R}^6$.
Thm For all $\mathrm{COL}\colon \mathbb{R}^5 \to [2]$ there exists a Mono Unit Square.

What About $\mathbb{R}^5$?

Observation The 15 vectors
$p_{1,2} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0, 0, 0)$, $p_{1,3} = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0, 0, 0)$, $\cdots$,
$p_{5,6} = (0, 0, 0, 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.
Span a 5-dimensional subspace H of $\mathbb{R}^6$.
Thm For all $\mathrm{COL}\colon \mathbb{R}^5 \to [2]$ there exists a Mono Unit Square.
Use that a coloring of $\mathbb{R}^5$ can be viewed as a coloring of H and then use the proof we did for $\mathbb{R}^6$.

What About $\mathbb{R}^4$?

Note that the proof we presented for $\mathbb{R}^6$ used very little geometry.

What About $\mathbb{R}^4$?

Note that the proof we presented for $\mathbb{R}^6$ used very little geometry.

Kent Cantwell showed

What About $\mathbb{R}^4$?

Note that the proof we presented for $\mathbb{R}^6$ used very little geometry.

Kent Cantwell showed
Thm For all $\mathrm{COL}\colon \mathbb{R}^4 \to [2]$ there exists a Mono Unit Square.

What About $\mathbb{R}^4$?

Note that the proof we presented for $\mathbb{R}^6$ used very little geometry.

Kent Cantwell showed
Thm For all $\mathrm{COL}\colon \mathbb{R}^4 \rightarrow [2]$ there exists a Mono Unit Square.
His proof used a lot more geometry than the proof for $\mathbb{R}^6$ and $\mathbb{R}^5$.

What About $\mathbb{R}^4$?

Note that the proof we presented for $\mathbb{R}^6$ used very little geometry.

Kent Cantwell showed
Thm For all $\mathrm{COL}\colon \mathbb{R}^4 \to [2]$ there exists a Mono Unit Square.
His proof used a lot more geometry than the proof for $\mathbb{R}^6$ and $\mathbb{R}^5$.
Here is the link to the paper:
`https://www.cs.umd.edu/~gasarch/COURSES/752/S25/slides/R4square.pdf`