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Finite Euclidean Ramsey Theory

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Let K be a configuration, a set of points in some finite dimensional Euclidean space. Let n and k be positive integers. The notation $R(K, n, r)$ is an abbreviation for the following statement: For every r-coloring of the points of n-dimensional Euclidean space, R^n, there exists a monochromatic configuration L which is congruent to K. In this paper, it is shown that when K is a square of side l, it can be proved that $R(K, 4, 2)$ holds. When K consists of two points at unit distance, it is also proved that $R(K, 4, 6)$ and $R(K, 5, 8)$ hold. © 1996 Academic Press, Inc.

1. INTRODUCTION

In [5], Erdős *et al.* proved the following theorem:

THEOREM 1.1 [5]. *If R^6 is 2-colored, then a monochromatic unit square is formed.*

Later a trivial change in the proof gave that if R^5 is 2-colored a monochromatic unit square is formed.

In this paper, we prove the following result in finite Ramsey theory.

1. If R^4 is 2-colored, a monochromatic unit square occurs (Theorem 2.10).

The chromatic number of R^n is the least number k so that R^n can be divided into k subsets with none of the subsets containing two points at unit distance.

THEOREM 1.2 [23]. *For $n=2, 3, 4$ and 5, the chromatic number of R^n is known to have lower bounds 4, 5, 6 and 8 respectively.*

An asymptotic upper bound of the form $(3+o(1))^n$ for the chromatic number of R^n was given by Larman and Rodgers [23]. A lower bound given by Frankl and Wilson [13] is $(1+o(1))(1.2)^n$.

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We will give improved bounds for $n=4$ and 5 by geometric methods. The new lower bounds when $n=4$ and $n=5$ will be 7 (Theorem 3.1) and 9 (Theorem 4.1), respectively. The proof in the case $n=4$ will use the result that a 2-coloring of 4-dimensional space gives a monochromatic square.

2. Any 2-Coloring of 4-Dimensional Space Must Contain a Monochromatic Square of Side a

Erdős *et al.* [5] have shown that:

THEOREM 2.1 [5]. *For any b, if R^6 is 2-colored, a monochromatic square of side b results.*

Proof. We include their proof, as it is relevant to our own. They use the fact that if the complete graph on 6 points is two-colored, it must contain a monochromatic four cycle (we are assuming an edge coloring). They then look at all 15 six dimensional points that have two coordinates equal to $b/(2^{1/2})$, and the rest of the coordinates equal to zero. They induce a coloring of the complete graph on six points from the coloring of these points by assigning the edge ac the color of the point with non-zero coordinates at positions a and c. Then an easy set of computations reveals than the resultant monochromatic four cycle corresponds to a monochromatic square whose side is b. Later, it was realized this actually proved that a two-coloring of 5-dimensional space must result in a monochromatic square; as the set of 15 points all lie in a hyperplane.

Call the set of points in 5-dimensional space with two coordinates equal to some constant $b/(2^{1/2})$ and the rest of the coordinates equal to 0 the *standard configuration* of side b. This configuration can be embedded in four dimensional space because all its points lie in a hyperplane–the sum of the coordinates is always $(2^{1/2})b$. Form an induced coloring of the complete graph on 5 points as follows: the edge ij is given the color of the point of the standard configuration of side b with non-zero coordinates i and j. We note some correspondences for future reference:

LEMMA 2.2. *Define mapping p as sending the edge ij of the complete graph on 5 points to the point of the standard configuration of side b with non-zero coordinates at i and j. Since p is one to one, we can extend to a mapping of sets of edges to sets of points by sending a set of edges, $\{E_i\}$, to the set of images, $\{p(E_i)\}$. We say that a set of edges corresponds to a set of points if p sends the set of edges to the set of points. Note that all the* *geometric shapes below have sides of length b and when we speak of a geometric configuration we mean its vertices.*

(1) *Four edges adjacent to a point correspond to a regular tetrahedron;*

(2) *A 4-cycle corresponds to a square;*

(3) *A 3-cycle corresponds to an equilateral triangle;*

(4) *Three edges adjacent to one point corresponds to an equilateral triangle;*

(5) *The complete graph on 4 points corresponds to an octahedron; and*

(6) *The complete graph on 4 points with one edge missing corresponds to the square-based pyramid.*

LEMMA 2.3. *There are only two subgraphs of the complete graph on 5 points which have the property that neither they nor their complement contain a four cycle: These two subgraphs are the 5 cycle, and the graph with 5 edges*: 12, 23, 13, 24 *and* 35.

LEMMA 2.4. *If 4-dimensional space is 2-colored in a manner so that it does not contain a monochromatic square of side b, then it also does not contain a monochromatic tetrahedron of side b.*

Proof. If the 4-dimensional space contains a monochromatic tetrahedron K of side b, then place a congruent copy of the standard configuration with side b so four vertices adjacent to one point correspond to the vertices of K. Then since the graph is not one of the two types described in Lemma 2.3, it will contain a monochromatic four cycle which will correspond to a square of side b, resulting in a contradiction.

LEMMA 2.5. *Consider a 2-coloring of 4-space which does not contain a monochromatic square with side of length b. In that 2-coloring, let T_1 and T_2 be two monochromatic equilateral triangles with different colors. If the planes determined by T_1 and T_2 are parallel and perpendicular to the line segment L their centroids, then the length of L is greater than $2(2^{1/2}/3^{1/2})\,b = c$.*

Proof. The two triangles both have circles of radius $c/2$ corresponding to those points which together with the triangles form a tetrahedron of side b ($c/2$ is the altitude of a tetrahedron of side b). If the conclusion of the lemma does not hold, these circles intersect at a point and if this point is either color, it forms a monochromatic tetrahedron of side b with one of the triangles. This yields a contradiction by Lemma 2.4.

LEMMA 2.6. *The radius of the circle of points which are distance b away from each of the endpoints of the two segments which are translates of each other by a vector orthogonal to the lines which contain the line segments is a continuous function* $((3/4)\,b^2-(1/4)\,d^2)^{1/2}$ *of the length of the translating vector. So is the magnitude in radians of the angle* $\theta = 2\sin^{-1}(b/(2((3/4)\,b^2-(1/4)\,d^2)^{1/2}))$ *on the circle between two points on the circle distance b apart.*

Proof. We note that on the plane containing the two line segments the midpoint, m, of the centers of the segments is the only point equidistant from all the endpoints of the segments. Its distance from each of the four points is easily seen to be $(d^2+b^2)^{1/2}/2$. Now assume we have a point, p, distance b from each of the endpoints. Then p and the four endpoints determine a three-space in the four-space. In three-space we know that the set of points equidistant from the vertices of a rectangle is the line through the centroid of the rectangle perpendicular to the plane of the rectangle. In this case the centroid of the rectangle is m. Using the Pythagorean theorem we see the distance from p to m must be $((3/4)\,b^2-(1/4)\,d^2)^{1/2}$. In 4-space it can easily be seen that the set of points a fixed distance from a point and having the property that the line they form with that point is perpendicular to a certain plane form a circle. Clearly the radius of that circle must be $((3/4)\,b^2-(1/4)\,d^2)^{1/2}$ and we have proven the first part of the theorem.

To prove the second part, recall that the formula for the length of a chord whose angle is θ is $2r\sin(\theta/2)$. So let us set $b=2r\,r\sin(\theta/2)$ then $b/2r=r\sin(\theta/2)$ and $2\sin^{-1}(b/2r)=\theta$. Finally, substituting $((3/4)\,b^2-(1/4)\,d^2)^{1/2}$ for r we get $\theta=2\sin^{-1}(b/((2((3/4)\,b^2-(1/4)\,d^2)^{1/2}))$.

LEMMA 2.7. *Consider a 2-coloring of 4-dimensional space which does not contain a monochromatic square of side b. In such a coloring, there is a dense set S of distances between 0 and $(2^{1/2})\,b$ such that whenever two parallel line segments of length b in the coloring have monochromatic endpoints with distinct colors and the perpendicular bisector of both segments coincide, then the distance between the centroids of the two segments is not a member of S.*

Proof. If the above conditions hold, there is a circle of points which are distance b away from each of the endpoints of the two segments. From the previous lemma, the radius of the circle is a continuous function $((3/4)\,b^2-(1/4)\,d^2)^{1/2}$ of the distance between the line segments. We note that within the range of values between 0 and $(2^{1/2})\,b$ the diameter is $>b$. This means it is meaningful to talk in this range of the magnitude in radians of the angle on the circle between two points on the circle distance b apart. This magnitude, $\theta=2\sin^{-1}(b/(2((3/4)\,b^2-(1/4)\,d^2)^{1/2}))$. Then the set of distances that cause such an arc length of the form $2\pi(d/e)$ where d and e are odd integers, is dense within the range of values between 0 and

$(2^{1/2})\, b$. In a circle where this occurs, there is an odd set of points which can be numbered from 1 to $2n+1$ such that i and $i+1$ are distance b apart, and 1 and $2n+1$ are distance b apart. Given any 2-coloring, there will be two points in this set at distance b apart and having the same color. These points together with one of the segments will form a monochromatic tethrahedron. This is a contradiction.

COROLLARY 2.8. *Consider a 2-coloring of 4-dimensional space which does not contain a monochromatic square with side length b. Let s_1 be a member of S, the set in Theorem 3.1.7. In this 2-coloring, there is no monochromatic equilateral triangle R_1 with side b which is a translate in a direction perpendicular to its plane by vector of length s_1 from an equilateral triangle R_2 of side b, with two vertices the opposite color from the color of R_1.*

Proof. We obtain a contradiction by considering the two vertices of opposite color in R_2 and the corresponding two vertices of R_1. These 4 points satisfy the hypothesis of the preceding lemma.

LEMMA 2.9. *If a coloring does not contain a monochromatic square of side b, and if A_1 and A_2 are two monochromatic equilateral triangles of side b in that coloring of the same color which are translates of each other in a direction perpendicular to their planes, then the distance of the translating vector must be greater than $(2^{1/2})\, b$.*

Proof. We will construct a plane P consisting of points which correspond to translates of the original triangle, in a direction perpendicular to the plane of the original triangle as follows: We let the centroid of each triangle represent it. Call the two centroids corresponding to the original two triangles a_1 and a_2. Call the set of circles in P centered at a_1 with radius a member of S, Δ. Because S is dense, there will be a circle δ in Δ which will be intersected by the circle λ of points in P distance b from A_2. We need density of S in case a_2 is very close to a_1. Then we use it to find a member of S, w such that $b < w < b + |a_1 a_2|$. Then the circle δ with radius w about a_1 will intersect λ.

Let q denote a point of intersection of δ with λ. Then the triangle Q associated with q is a translate of the triangle A_2 by vector aq which has length b. So if Q has two vertices the color of the original triangle, A_2, they will form, together with the corresponding vertices in A_2 a monochromatic square of side b. So Q must have at least two points of opposite color of A_2. But these two points together with the corresponding points in A_1 will form two segments which are translates by a vector of length w. Since w is a member of S, this contradicts Lemma 2.7.

Before proceeding any further let us note that a four dimensional cross polytope is the four dimensional analog of the octahedron and can be realized by the points $(\pm 1, 0, 0, 0)$, $(0, \pm 1, 0, 0)$, $(0, 0, \pm 1, 0)$ and $(0, 0, 0, \pm 1)$.

LEMMA 2.10. *If a coloring does not contain a square of side b, then in that coloring, each four dimensional cross polytope of side b, then in that coloring, each four dimensional cross polytope of side b must have exactly* 4 *monochromatic triangles among the* 32 *equilateral triangles of side b its vertices form.*

Proof. Consider the four pairs of points which are $(2^{1/2})\,b$ apart from each other. If 3 pairs or more are dichromatic, then they form two monochromatic equilateral triangles of side b which are of opposite colors and satisfying the conditions of Lemma 2.5. Then by Lemma 2.5 a monochromatic square of side b is formed and we get a contradiction.

If one or less pair is dichromatic, we will have 3 or more monochromatic pairs. Then two of the three or more monochromatic pairs form a monochromatic square of side b which results in a contradiction.

So exactly two of the pairs are dichromatic. If both remaining pairs are the same color, a monochromatic square is formed which again results in a contradiction. So the remaining two pairs are monochromatic of opposite colors, and hence we have fixed one coloring for all four pairs. Thus we have fixed a coloring for the 4-dimensional cross polytope. This coloring will result in exactly four of the 32 equilateral triangles formed by the vertices being monochromatic.

THEOREM 2.11. *If* 4*-dimensional space is* 2*-colored, then a monochromatic square with side b is formed.*

Proof. Consider all four dimensional cross polytopes with side of length b, with centers at $(n_1/m100000, n_2/m100000, n_3/m100000, n_4/m100000)$ with all n_i between 0 and m whose vertices consist of the center translated by a distance equal to $b/(2^{1/2})\,b$ in each of the 8 directions parallel to one of the coordinate axes. Now for each of the 32 possible triangles in a four dimensional cross polytope of this type there are at most m^3 monochromatic triangles:

For each triangle, there is at least one coordinate whose direction is orthogonal to the plane of the triangle. We note that we have a set of 32 equivalence classes of triangles each class containing triangles which are translates of each other. Then for each two triangles which are translates of each other with the three coordinates equal whose axes are non-orthogonal to the plane of the triangles, both of them cannot be monochromatic or there will be a contradiction with Lemma 2.5 or Lemma 2.8.

Hence we have at most $32m^3$ monochromatic triangles. By Lemma 2.9 there must be $4m^4$ monochromatic triangles (4 for each octahedron). This results in a contradiction if $m > 1000$. So we must have a monochromatic square with side b and we are done.

3. The Chromatic Number of 4-Dimensional Space Is at Least 7

Recall that the chromatic number of n-space is the least positive integer t such that the points of R^n can be partitioned into t subsets with no two points in any subset at unit distance. Clearly, this definition does not depend on "unit" distance, and we may instead insist that no two points are in any subset at distance b, where b is any fixed positive number.

The primary goal of this section will be to prove the following lower bound.

THEOREM 3.1. *The chromatic number of* 4-*space is at least* 7.

Proof. The proof of this theorem will require a series of lemmas. Throughout this section we assume that we have a fixed 6-coloring, of 4-space and that no two points of the same color are at distance 1. We argue to a contradiction. First, we combine the six colors into two groups of three.

LEMMA 3.2. *If under the resultant 2-coloring, there is no resultant monochromatic pyramid of side b, there is a regrouping of the six colors into two new groups of three, such that one of the two new groups contains a monochromatic pyramid.*

Proof. Assume we have a 2 coloring resulting from two groups of three colors each. By the result of the previous section, we have a monochromatic square of one color of side b. We now imbed this monochromatic square of side b in a monochromatic cross polytope of side b. Each vertex of the cross polytope outside the square forms a pyramid with the square. Then since there is no monochromatic pyramid, the four vertices of the cross polytope besides the square must be opposite color to the square. Hence the cross polytope consists of two monochromatic squares of opposite colors.

Now since each group contains three colors, each square must contain two points of the same color. Call the points in the first square of the same color c_1 and c_2, and the points in the second square of the same color d_1 and d_2. Now c_1 and c_2 cannot be distance b apart because they are the same color. So they must be distance $(2^{1/2})\, b$ apart. By a similar proof, d_1 and d_2 must be $(2^{1/2})\, b$ apart. Using this information we can easily see c_1, c_2, d_1, and d_2 form a square of side b. Together with a third point from the cross polytope, they form a pyramid with at most three original colors. Form a new grouping with this group of three in one group (if it is a group of two add an original color at random) and let the other group contain the other colors. Thus we have a new set of two groups each containing three colors such that one group forms a monochromatic pyramid of side b.

We use Lemma 3.2 to find a division of the 6-colors into two groups of three colors each such that one group contains the points of a pyramid of side b. This division into two groups will be the 2-coloring referred to in what follows. We say a configuration is monochromatic if all its points are in the same group. We divide the proof into two cases depending on whether or not one color contains the vertices of an octahedron of side b.

Case 1. A monochromatic octahedron of side b is formed.

Without loss of generality, let the octahedron be red. We imbed it in a standard configuration (defined in Section 2) so that it can be put into a correspondence with the complete graph on 4 points. From now on when we refer to the color of an edge we mean the color of the point it corresponds to under the correspondence of Lemma 3.2. Recall that under this correspondence the complete graph on 5 points corresponds to the standard configuration. Then note that none of the edges which connect the fifth point to any of the four points in the complete graph can be red, or we would have four red edges adjacent to the same point. This would give us a red tethrahedron of side b. In this case, two of the four points of the tetrahedron would have to be of the same original color, and we would get two points of the same original color distance b apart. This is a contradiction. So all the edges connecting the fifth point to the four points must be blue. We then obtain four blue edges adjacent to one point. This yields a blue tetrahedron of side b, which results in the same contradiction as above.

Case 2. There is no monochromatic octahedron of side b.

By Lemma 3.2 we have a monochromatic pyramid. Without loss of generality let it be red. Recall the correspondence between the complete graph on 5 points and the standard configuration. Imbed the pyramid in the standard configuration so that it forms a complete graph on four points with one edge removed. Label the points with the edge removed between them 1 and 4. Label the other two points in the complete graph 2 and 3. Label the remaining point 5. The edges 52 and 53 cannot be red, or we would get four edges of the same color at one point. This implies a monochromatic tethrahedron, and the same contradiction reached in Case 1. So edge 52 and 53 must be blue.

If edge 54 were blue then the complete graph on the points 2, 3, 4 and 5 would correspond to an octahedron of side b with all 3 pairs of its opposite points dichromatic. If edge 51 were blue then the complete graph on the points 1, 2, 3 and 5 would correspond to an octahedron of side b with all 3 pairs of its opposite points dichromatic. We have shown in Section 2 that an octahedron of side b with all 3 pairs of its opposite points dichromatic leads to a monochromatic tetrahedron of side b. But this leads to a contradiction as noted earlier. So both edges 54 and 51 must be red. Now we have obtained a coloring of all the edges of the complete graph on 5 points.

The resultant set of red edges, when three colored must contain two adjacent edges. To see this, let the three colors be purple, blue, and green. Now without loss of generality let edge 23 be purple. Then edges 24, 43, 31, and 12 are all adjacent to 23 and hence must be blue or green. Since edges 24 and 43 are adjacent one must be blue and one must be green. Since 54 is adjacent to both 24 and 43 it must be purple. Since 31 and 12 are adjacent one must be blue and one must be green. Since 51 is adjacent to both 31 and 12 it must be purple. But 54 and 51 cannot both be purple because they are adjacent. This gives us a contradiction.

So we must have two adjacent edges with the same color. These edges correspond to points distance b apart that have the same original color. This contradiction completes the proof of Theorem 3.1.

4. The Chromatic Number of R^5 Is at Least 9

In this section, we prove the following lower bound.

Theorem 4.1. *The chromatic number of* 5-*space is at least* 9.

Again, because it is central to our approach we begin with the proof of a weaker result.

Theorem 4.2 [21]. *The chromatic number of R^5 is at least* 8.

Proof. If we wish to show that the distance $(2^{1/2})$ must occur in any 7-coloring, we look at the configuration consisting of all points having an even number of coordinates equal to 1 and the rest of the coordinates equal to zero. This configuration, called the half cube, contains 16 points: One with all zero's, 10 with 2 non-zero coordinates, and 5 with 4 non-zero coordinates. Now all distances between the points of the half cube are 2 and $(2^{1/2})$. Moreover no 3 points can have distances all equal to 2 as then the minimum sphere they could lie in would have radius equal to $2/(3^{1/2})$ but the entire half cube can be imbedded in a sphere of radius $(5^{1/2})/2$, centered about the point with all coordinates equal to 1/2 which gives a contradiction. So we have there must be at most two elements of each color in the half cube. Since we have 16·elements, we need at least 8 colors.

We further note that in any 8 coloring of R^5, the half cube must have exactly two elements of each color.

DEFINITION 4.3. Call the configuration consisting of all points whose coordinates are the sum of the coordinates of any two points of the half cube the half cube sum.

To illustrate what this is, let us look at its points: First, since the origin is in the half cube the original half cube is in the half cube sum. Any point in the half cube added to itself gives the original point with its coordinates multiplied by 2. It also contains 30 points with two 0 coordinates, two 1 coordinates, and one 2 coordinate; 30 points with two 2 coordinates two 2 1's and one 0; 10 points with two 1's and three 2's, and 5 points with one 2 and four 1's.

DEFINITION 4.4. The half cube sum contains configurations isomorphic to the half cube. For each element of the half cube, there is a configuration consisting of the sum of each element of the half cube with that element. For each element a, we designate this configuration as the a-half cube. Thus the original half cube would be designated the 0-half cube. From the above discussion in an 8-coloring with no two monochromatic points distance $(2^{1/2})$ apart, each a-halfcube must contain exactly two elements of each color class.

LEMMA 4.5. *In a 8-coloring with no two monochromatic points at distance $(2^{1/2})$ apart when the points $(0, 0, 0, 0, 0)$ and $(1, 1, 1, 1, 2)$ are the same color, then $(1, 1, 1, 1, 0)$ is that color.*

Proof. We must have another point that color in the $(0, 0, 0, 0, 0)$-half cube. However, that second point cannot have two 1's or a pair of points distance $(2^{1/2})$ apart is formed. So it must have four 1's but if it is not $(1, 1, 1, 1, 0)$ it is at distance $(2^{1/2})$ from $(1, 1, 1, 1, 2)$. Hence it must be $(1, 1, 1, 1, 0)$.

LEMMA 4.6. *If we have an 8-coloring with no monochromatic points distance $(2^{1/2})$ apart, two points c and d of the same color cannot be distance $2(2^{1/2})$ apart.*

Proof. If they were, the set of points S at distance 2 from c and d must be the same color as c and d. This follows from Lemma 4.5 for a suitable half cube. But then the points in S would form a monochromatic 4-dimensional sphere of radius $(2^{1/2})$, which would contain 2 points of the same color at distance $(2^{1/2})$.

COROLLARY 4.7. *Given an* 8*-coloring with no monochromatic points distance* $(2^{1/2})$ *apart, if two points with two coordinates equal to* 2 *and the rest equal to* 0 *are the same color, then they cannot both have a* 2 *in the same coordinate position.*

Proof. If they do, the distance between them is $2(2^{1/2})$, and we have a contradiction by Lemma 4.6.

COROLLARY 4.8. *Given an* 8*-coloring with no monochromatic points distance* $(2^{1/2})$ *apart, if a point* x *with one coordinate equal to* 2 *and the rest equal to one is the same color as a point* y *with two coordinates equal to* 2 *and the rest equal to zero, then the coordinate of* x *which is* 2 *is it the same as the coordinate of* y *which is* 2.

Proof. If not, then the distance between the two points is $2(2^{1/2})$, and we get a contradiction by Lemma 4.6.

COROLLARY 4.9. *Given an* 8*-coloring with no monochromatic points distance* $(2^{1/2})$, *apart there cannot be three points of the same color with two coordinates equal to* 2 *and the rest* 0.

Proof. The three points would have six coordinates equal to 2 between them. Hence, for one coordinate position there must be two different points with the value 2. This leads to a contradiction with Corollary 4.7.

Proof of Theorem 4.1. Assume that we have an 8-coloring of 5-space and no two points of the same color are at unit distance. We argue to a contradiction. The five points with four 1's and one 2 and the point with all coordinates equal to 0 must be six different colors. If two are the same color, then the distance between them is either $2(2^{1/2})$ or $(2^{1/2})$, and in either case we get a contradiction.

We have 10 points G with two coordinates equal to 2 and the rest equal to 0. None of these points can be the same color as point $(0, 0, 0, 0, 0)$, because they would be distance $2(2^{1/2})$ apart. For each point with four 1's and one 2, there can only be one point in this group with the same color by Lemma 4.8. Only five points of G can be in the six colors mentioned in the above paragraph. This gives a total of at most five points in these six groups. Then we must have at least 5 points of this type in the remaining 2 colors. Then we must have three points in one color, but that gives a contradiction by Lemma 4.9 and we are done.

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