$\mathbf{PH(1) \le 7}$

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Notation We call a set like H a **Large Homog Set** and abbreviate this by **LHS**.

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Assume We can assum $\mathrm{COL}(1,2) = R$.

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- $(\forall 1 \le i < j \le 3)[\mathrm{COL}(x_i, x_j) = B]$. LHS: $\{2, x_2, x_3\}$

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Last Case on Next Slide.

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$(\forall 1 \le i < j \le 4[\mathrm{COL}(x_i, x_j) = B \wedge x_1 \ge 4]$.
1) $x_1 \ge 5$. Then $x_2 \ge 6$, $x_3 \ge 7$, $x_4 \ge 8$. Contradiction.

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  Map $i \in \{4, 5, 6, 7\}$ to $j \in \{2, 3\}$ st $\mathrm{COL}(i, j) = R$.

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  $(\exists i, j \in \{4, 5, 6, 7\})$ map to $2 \rightarrow$ LHS $\{2, i, j\}$.
  $(\exists i, j, k \in \{4, 5, 6, 7\})$ map to $3 \rightarrow$ then LHS $\{3, i, j, k\}$.

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  $(\exists i, j \in \{4, 5, 6, 7\})$ map to 2 $\rightarrow$ LHS $\{2, i, j\}$.
  $(\exists i, j, k \in \{4, 5, 6, 7\})$ map to 3 $\rightarrow$ then LHS $\{3, i, j, k\}$.
  If neither happens then ≤ 1 element of $\{4, 5, 6, 7\}$ maps to 2 and ≤ 2 elements of $\{4, 5, 6, 7\}$ map to 3. So ≤ 3 elements get mapped, contradiction.

Case 3: NOT Case 1 or 2

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So $\deg(1) \leq 5$ which is a contradiction.