The Large Ramsey Theorem

Exposition by William Gasarch

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3. 2^A is the powerset of A.
4. $\binom{A}{a}$ is the set of all a-sized subsets of A.

Let $\mathrm{COL}\colon \binom{A}{2} \rightarrow [2]$. A set $H \subseteq A$ is **homogenous** if COL restricted to $\binom{H}{2}$ is constant. (From now on **homog**.)

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$\{5, 30, 40, 50, 60, 70, 80, 90, 100\}$ is large.

$\{101, \ldots, 190\}$ is NOT large.

Infinite And Finite Ramsey Thm

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This is a stupid thm. Discuss.
$\{1, 2\}$ is always a large homogenous set.
How to change the statement so thats its not stupid?

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$\min(H) = 10$ and $|H| \geq 11$, or

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$\min(H) = 499$ and $|H| \geq 500$, or

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$\min(H) = 499$ and $|H| \geq 500$, or
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$\min(H) = 499$ and $|H| \ge 500$, or
$\min(H) = 500$ and $|H| \ge 501$. $H = \{500, \ldots, 1000\}$.
If $\min(H) = 501$ then H cannot be large.

Proof of the Large Ramsey Thm From The Infinite Ramsey Thm

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The proof will be identical to the proof of

$$\textbf{Infinite Ramsey} \implies \textbf{Finite Ramsey}$$

except at the very end.

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Say $k = 182$. There is a coloring of $\binom{\{182,\ldots,182+10^{100}\}}{2}$ with no large homog set.

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Say $k = 182$. There is a coloring of $\binom{\{182,\ldots,182+10^{100}\}}{2}$ with no large homog set. That seems unlikely.

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We use $\mathrm{COL}_0, \mathrm{COL}_1, \ldots$ to form
$\mathrm{COL}\colon \binom{\mathbb{N}}{2} \rightarrow [2]$.
We will use the **Inf Ramsey Theory** to get a contradiction.

Forming COL

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Assume $\mathrm{COL}(e_1)$, ..., $\mathrm{COL}(e_s)$, I_{s+1} are defined.
$\mathrm{COL}(e_{s+1}) =$ **R** if $|\{y \in I_{s+1} \colon \mathrm{COL}_y(e_{s+1}) =$ **R**$\}| = \infty$, **B** OW.

$I_{s+2} = \{y \in I_{s+1} \colon \mathrm{COL}_y(e_{s+1}) = \mathrm{COL}(e_{s+1})\}$

Using COL To Get a Contradiction

We have defined $\mathrm{COL}\colon \binom{\mathbb{N}}{2} \to [2]$.

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By **The Infinite Ramsey Thm** there exists infinite homog set

$$H = \{x_1 < x_2 < x_3 < x_4 < \cdots\}$$

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Continue on Next Slide.

Finite Ramsey VS Large Ramsey

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Proof of Large Ramsey: The following is a large homog set:

$$H_{\mathrm{LargeSet}} = \{x_1 < \cdots < x_{x_1+1}\}.$$

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Proof of Large Ramsey: The following is a large homog set:

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There is an L such that COL restricted to H_{LargeSet} is some COL_L.

Finite Ramsey VS Large Ramsey

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Proof of Large Ramsey: The following is a large homog set:

$$H_{\rm LargeSet} = \{x_1 < \cdots < x_{x_1+1}\}.$$

There is an L such that $\rm COL$ restricted to $H_{\rm LargeSet}$ is some COL_L.

This is a contradiction since COL_L has no large homog sets.

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STUDENT: You are telling the same jokes twice, with **Shirley** and **Factorial**.

Twice

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Who first said

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Answer on next slide.

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Discuss: Is LR natural?