History of Upper Bounds On: VDW Numbers

Exposition by William Gasarch

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Recall VDW's Theorem

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there exists a, d such that

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Erdös and Turan had an idea!

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Erdös-Turan Conjecture For all $\delta > 0$, for all k, there exists $N = N(\delta, k)$ such that the following holds:

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Hope The proof of the ET-conj will be a diff proof of VDW's theorem that gives better bounds on the VDW numbers.

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7) 1988: The Hales-Jewitt Thm implies VDW's Theorem. Shelah gives a new proof of the HJ Thm which gives primitive recursive (though still quite large) bounds on the VDW numbers. The proof is elementary and does not use any of the ET stuff.

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$$W(k,c) \le 2^{2^{c^{2^{2^{k+9}}}}}.$$