Goodstein Sequences

Exposition by William Gasarch-U of MD

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This is called **Hereditary Base *n* Notation**

Ackerman's Function and Goodstein Seq

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Answer on Next Slide

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Now its a 2-digit number and use induction.

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Next Slide will indicate why am asking this.

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3. Harvey Friedman has done much research on this. Here is one of his theorems:
   `https://cpb-us-w2.wpmucdn.com/u.osu.edu/dist/1/1952/files/2014/01/FIiniteSeqInc062214a-v9w7q4.pdf`