Finite Ramsey Theorem For 3-Hypergraph

Exposition by William Gasarch

December 9, 2024

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(b) every $\{x_1, \ldots, x_a\} \in \binom{H}{a}$ has NOT written a paper together.

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We do some an example of the first few steps of the construction.

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What to make of this? Discuss.

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We are given $\mathrm{COL}\colon \binom{[n]}{3} \to [2]$.

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Key $|H_1| \geq \Omega(\log_2(n))$.

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Kill all those who disagree!

Construction of $x_1, H_1, c_1, x_2, H_2, c_2$

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Next Slide is General Case.

Construction of $x_{s+1}, H_{s+1}, c_{s+1}$

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Iterate this process $2k-1$ times.

The Coloring of the Nodes

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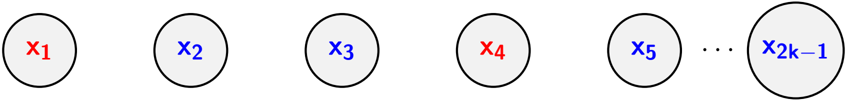

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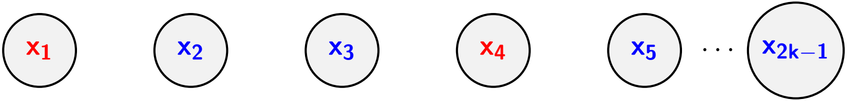

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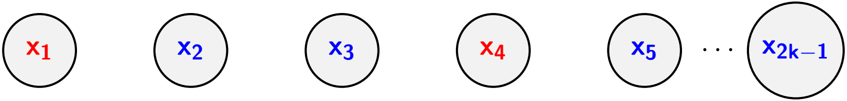

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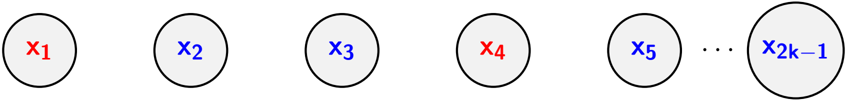

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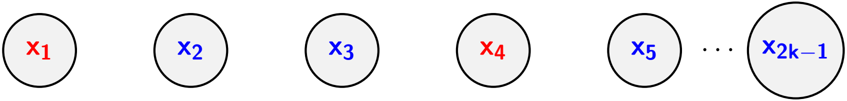

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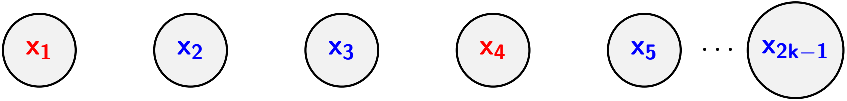

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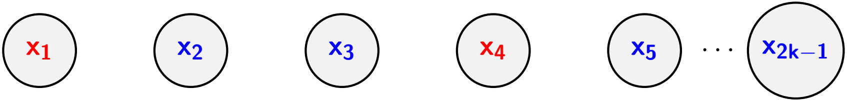
What do you think our next step is?

Some Color Appears k times

Some Color Appears *k* times



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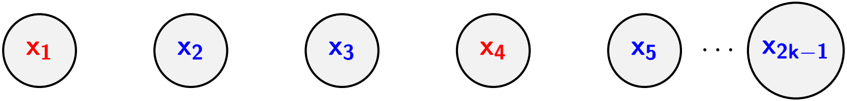

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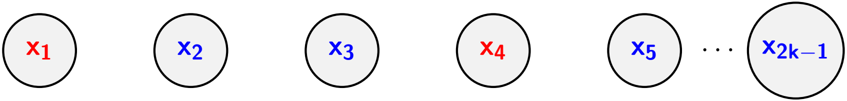

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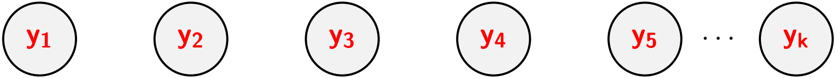

Some Color Appears k times

x_1 x_2 x_3 x_4 x_5 $\cdots$ x_{2k-1}

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y_1 y_2 y_3 y_4 y_5 $\cdots$ y_k

For all $i < j < k$, $\mathrm{COL}(x_i, x_j, x_k) = \mathbf{R}$. (More generally c.)

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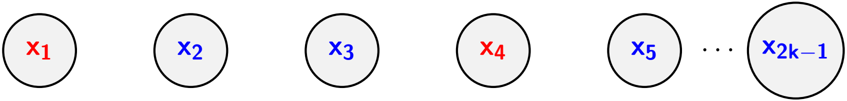

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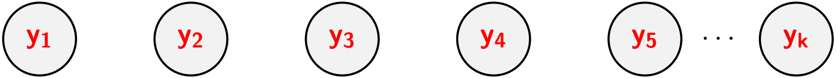

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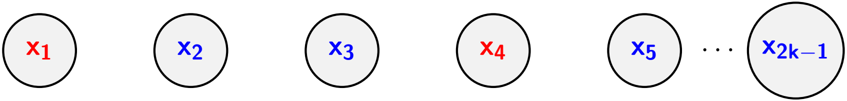

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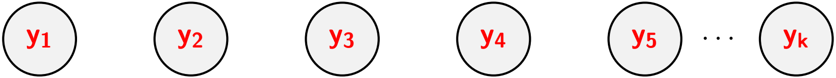

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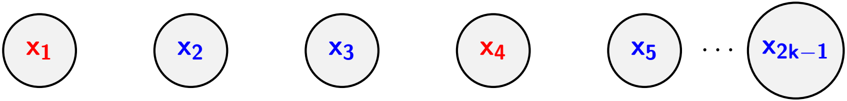

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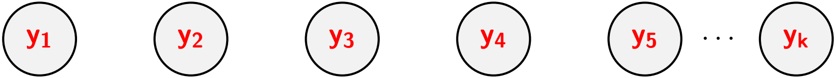

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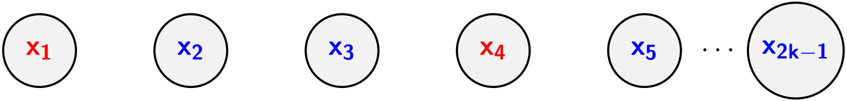

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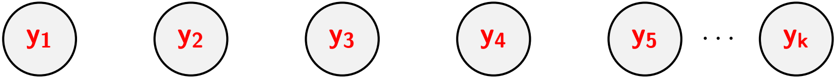

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What About 4-Hypergraph Ramsey Theory?

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Beyond that the functions have no name.

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Answer on the next slide.

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Conjecture There is no value of c such that $R_3(k) \leq 2^{ck}$.