Euclidean Ramsey Theory

Exposition by William Gasarch

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Ramsey Theory VS Euclidean Ramsey Theory

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In Euclidean Ramsey Theory we will be seek an object of a certain size, for example the unit square.

For All 2-Colorings of the Plane. . .

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Proof on the next page.

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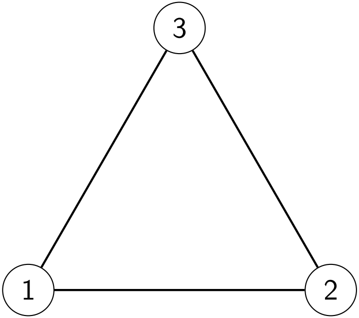

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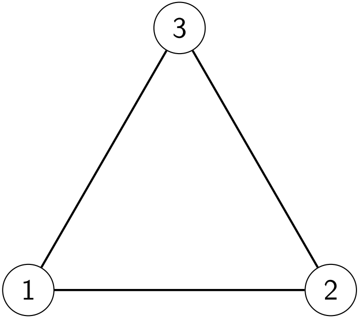

3 vertices and 2 colors. So 2 of the vertices are the same color.

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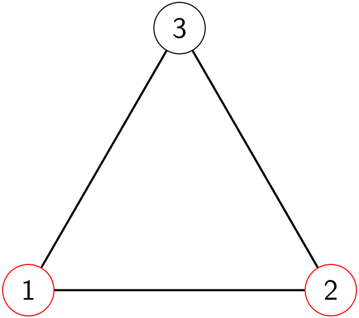

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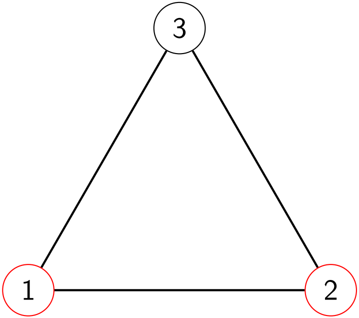

Vertices 1 and 2 are an inch apart.

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We investigate what χ can be.

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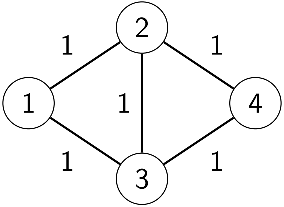

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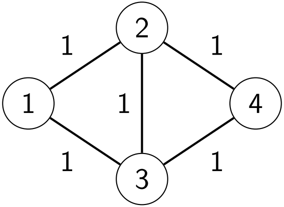

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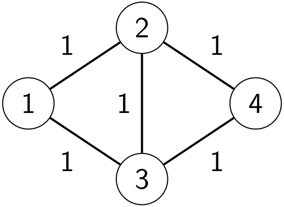

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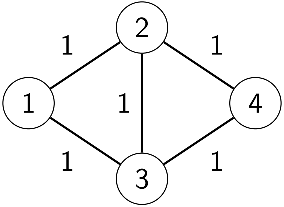

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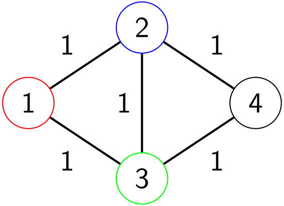

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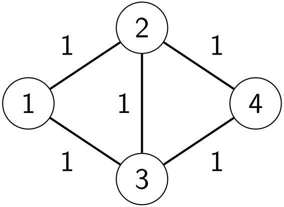

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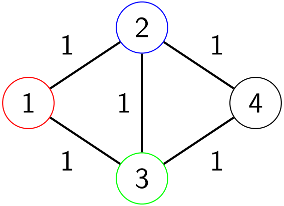

Hence $\mathrm{COL}(4) =$ **R**.

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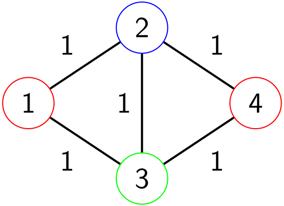

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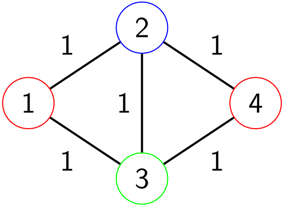

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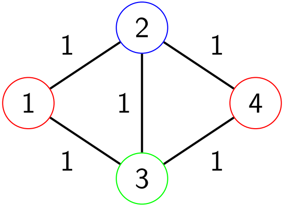

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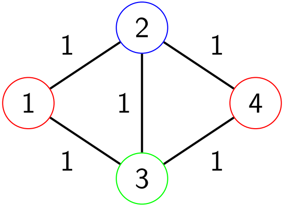

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Upshot 2 If $\mathrm{COL}(p) =$ **R** then circle of radius $\sqrt{3}$ around p is **R**.

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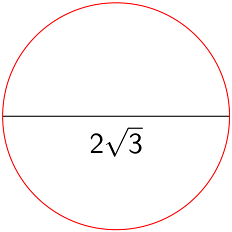

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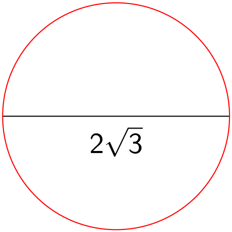

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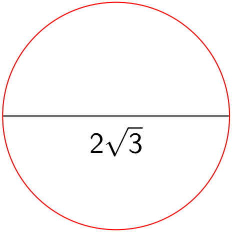

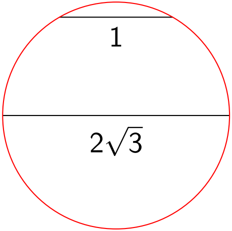

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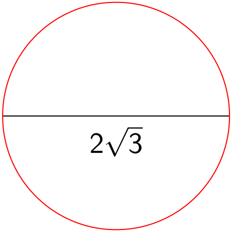

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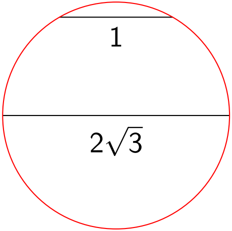

Endpoints of chord are **R** and an inch apart.

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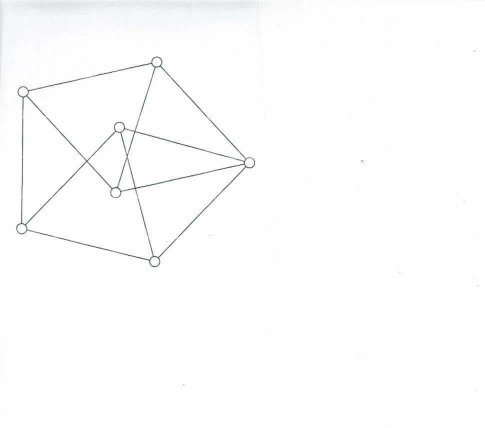

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So we know that $\chi \geq 5$.

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Here is the 7-coloring:
https://thatsmaths.com/2022/03/24/the-chromatic-number-of-the-plane/