Monochromatic C_4

Exposition by William Gasarch

November 20, 2024

Credit Where Credit is Due

The main theorem in these slides was proven by

Credit Where Credit is Due

The main theorem in these slides was proven by

Vaclav Chvtal and Frank Harary
in the paper

Credit Where Credit is Due

The main theorem in these slides was proven by

Vaclav Chvtal and Frank Harary
in the paper

Generalized Ramsey Theory for Graphs II: Small Diagonal Numbers

Credit Where Credit is Due

The main theorem in these slides was proven by

Vaclav Chvtal and Frank Harary
in the paper

Generalized Ramsey Theory for Graphs II: Small Diagonal Numbers

Proceedings of the American Mathematical Society. April 1972, Vol 32, No. 2, pages 389–394.

Credit Where Credit is Due

The main theorem in these slides was proven by

Vaclav Chvtal and Frank Harary
in the paper

Generalized Ramsey Theory for Graphs II: Small Diagonal Numbers

Proceedings of the American Mathematical Society. April 1972, Vol 32, No. 2, pages 389–394.

Here is a link
`https://www.cs.umd.edu/~gasarch/TOPICS/eramsey/c4.pdf`

When Do You Get A Mono C_4?

Thm For all $\mathrm{COL}\colon \binom{[18]}{2} \rightarrow [2]$ there exists a mono C_4.

When Do You Get A Mono C_4?

Thm For all $\mathrm{COL}\colon \binom{[18]}{2} \to [2]$ there exists a mono C_4.
Pf:

When Do You Get A Mono C_4?

Thm For all $\mathrm{COL}\colon \binom{[18]}{2} \to [2]$ there exists a mono C_4.

Pf:

Let $\mathrm{COL}\colon \binom{[18]}{2} \to [2]$.

When Do You Get A Mono C_4?

Thm For all $\mathrm{COL}\colon \binom{[18]}{2} \rightarrow [2]$ there exists a mono C_4.

Pf:

Let $\mathrm{COL}\colon \binom{[18]}{2} \rightarrow [2]$.

Since $R(4) = 18$ there is a mono K_4, hence a mono C_4.

When Do You Get A Mono C_4?

Thm For all $\mathrm{COL}\colon \binom{[18]}{2} \to [2]$ there exists a mono C_4.
Pf:
Let $\mathrm{COL}\colon \binom{[18]}{2} \to [2]$.
Since $R(4) = 18$ there is a mono K_4, hence a mono C_4.
End of Pf

When Do You Get A Mono C_4?

Thm For all $\mathrm{COL}\colon \binom{[18]}{2} \to [2]$ there exists a mono C_4.

Pf:

Let $\mathrm{COL}\colon \binom{[18]}{2} \to [2]$.

Since $R(4) = 18$ there is a mono K_4, hence a mono C_4.

End of Pf

$R(C_4)$ is the least n such that $\forall\ \mathrm{COL}\colon \binom{[n]}{2} \to [2]\ \exists$ mono C_4.

When Do You Get A Mono C_4?

Thm For all $\mathrm{COL}\colon \binom{[18]}{2} \rightarrow [2]$ there exists a mono C_4.
Pf:
Let $\mathrm{COL}\colon \binom{[18]}{2} \rightarrow [2]$.
Since $R(4) = 18$ there is a mono K_4, hence a mono C_4.
End of Pf

$R(C_4)$ is the least n such that $\forall\ \mathrm{COL}\colon \binom{[n]}{2} \rightarrow [2]\ \exists$ mono C_4.
We have shown $R(C_4) \leq 18$.

When Do You Get A Mono C_4?

Thm For all $\mathrm{COL}\colon \binom{[18]}{2} \to [2]$ there exists a mono C_4.
Pf:
Let $\mathrm{COL}\colon \binom{[18]}{2} \to [2]$.
Since $R(4) = 18$ there is a mono K_4, hence a mono C_4.
End of Pf

$R(C_4)$ is the least n such that $\forall\ \mathrm{COL}\colon \binom{[n]}{2} \to [2]\ \exists$ mono C_4.
We have shown $R(C_4) \le 18$.
Vote

When Do You Get A Mono C_4?

Thm For all $\mathrm{COL}\colon \binom{[18]}{2} \to [2]$ there exists a mono C_4.

Pf:

Let $\mathrm{COL}\colon \binom{[18]}{2} \to [2]$.

Since $R(4) = 18$ there is a mono K_4, hence a mono C_4.

End of Pf

$R(C_4)$ is the least n such that $\forall\ \mathrm{COL}\colon \binom{[n]}{2} \to [2]\ \exists$ mono C_4.

We have shown $R(C_4) \le 18$.

Vote

1) $R(C_4) = 18$.

When Do You Get A Mono C_4?

Thm For all $\mathrm{COL}\colon \binom{[18]}{2} \to [2]$ there exists a mono C_4.

Pf:

Let $\mathrm{COL}\colon \binom{[18]}{2} \to [2]$.

Since $R(4) = 18$ there is a mono K_4, hence a mono C_4.

End of Pf

$R(C_4)$ is the least n such that $\forall$ $\mathrm{COL}\colon \binom{[n]}{2} \to [2]$ $\exists$ mono C_4.

We have shown $R(C_4) \le 18$.

Vote

1) $R(C_4) = 18$.

2) $10 \le R(C_4) \le 17$.

When Do You Get A Mono C_4?

Thm For all $\mathrm{COL}\colon \binom{[18]}{2} \to [2]$ there exists a mono C_4.
Pf:
Let $\mathrm{COL}\colon \binom{[18]}{2} \to [2]$.
Since $R(4) = 18$ there is a mono K_4, hence a mono C_4.
End of Pf

$R(C_4)$ is the least n such that $\forall$ $\mathrm{COL}\colon \binom{[n]}{2} \to [2]$ $\exists$ mono C_4.
We have shown $R(C_4) \le 18$.
Vote
1) $R(C_4) = 18$.
2) $10 \le R(C_4) \le 17$.
3) $5 \le R(C_4) \le 9$.

When Do You Get A Mono C_4?

Thm For all $\mathrm{COL}\colon \binom{[18]}{2} \to [2]$ there exists a mono C_4.
Pf:
Let $\mathrm{COL}\colon \binom{[18]}{2} \to [2]$.
Since $R(4) = 18$ there is a mono K_4, hence a mono C_4.
End of Pf

$R(C_4)$ is the least n such that $\forall\ \mathrm{COL}\colon \binom{[n]}{2} \to [2]\ \exists$ mono C_4.
We have shown $R(C_4) \le 18$.
Vote
1) $R(C_4) = 18$.
2) $10 \le R(C_4) \le 17$.
3) $5 \le R(C_4) \le 9$.
Answer on the next page.

$R(C_4) = 6$

Thm $R(C_4) = 6$.

$R(C_4) = 6$

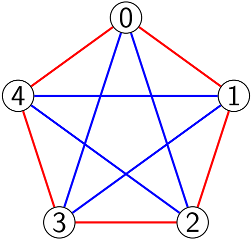
Thm $R(C_4) = 6$.
First Need $R(C_4) \geq 6$.

$R(C_4) = 6$

Thm $R(C_4) = 6$.
First Need $R(C_4) \geq 6$.
We present a $\mathrm{COL}\colon \binom{[5]}{2} \rightarrow [2]$ with no mono C_4.

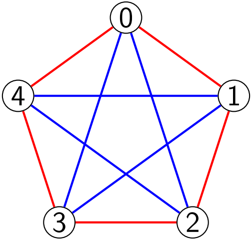
$R(C_4) = 6$

Thm $R(C_4) = 6$.
First Need $R(C_4) \ge 6$.
We present a $\mathrm{COL}\colon \binom{[5]}{2} \to [2]$ with no mono C_4.



$R(C_4) = 6$

Thm $R(C_4) = 6$.
First Need $R(C_4) \geq 6$.
We present a $\mathrm{COL}\colon \binom{[5]}{2} \rightarrow [2]$ with no mono C_4.



Note: There is a mono C_5 but not a mono C_4.

$R(C_4) = 6$

Second Need $R(C_4) \leq 6$.

$R(C_4) = 6$

Second Need $R(C_4) \le 6$.
We show that $\forall\ \mathrm{COL}\colon \binom{[6]}{2} \rightarrow [2]\ \exists$ mono C_4.

$R(C_4) = 6$

Second Need $R(C_4) \le 6$.
We show that $\forall\ \mathrm{COL}\colon \binom{[6]}{2} \rightarrow [2]\ \exists$ mono C_4.

Let $\mathrm{COL}\colon \binom{[6]}{2} \rightarrow [2]$.

$R(C_4) = 6$

Second Need $R(C_4) \le 6$.
We show that $\forall \ \mathrm{COL}\colon \binom{[6]}{2} \rightarrow [2] \ \exists$ mono C_4.

Let $\mathrm{COL}\colon \binom{[6]}{2} \rightarrow [2]$.

We know that there is a mono triangle.

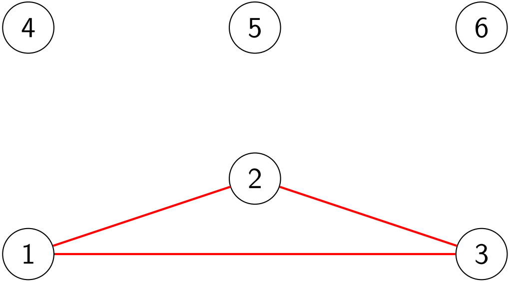
First Step

First Step

$\exists$ a mono triangle. We assume **R** and on vertices $\{1, 2, 3\}$.

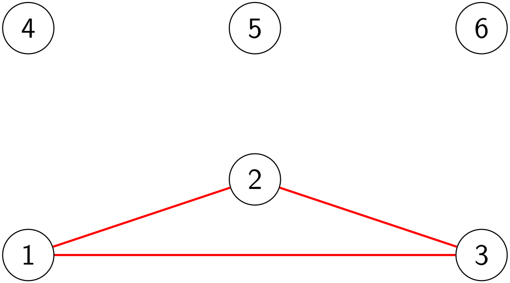
First Step

$\exists$ a mono triangle. We assume **R** and on vertices $\{1, 2, 3\}$.



First Step

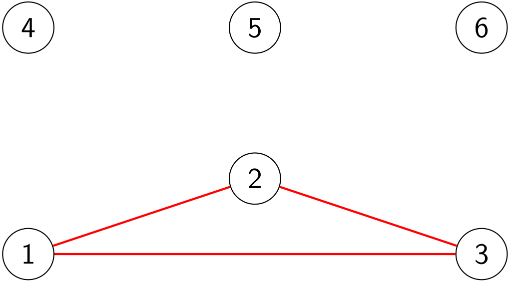
$\exists$ a mono triangle. We assume **R** and on vertices $\{1, 2, 3\}$.



We view $\{1, 2, 3\}$ and $\{4, 5, 6\}$ as the sides of a bipartite graph.

First Step

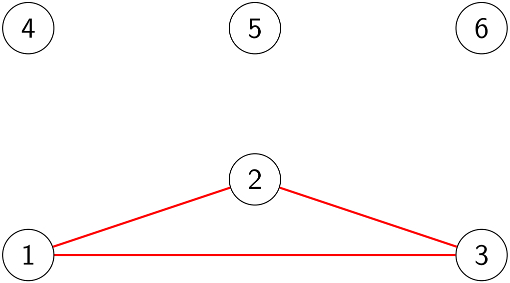
$\exists$ a mono triangle. We assume **R** and on vertices $\{1,2,3\}$.



We view $\{1,2,3\}$ and $\{4,5,6\}$ as the sides of a bipartite graph.
$\deg_{\mathbf{R}}(1)$ will mean the number of **R** edges between 1 and $\{4,5,6\}$.

First Step

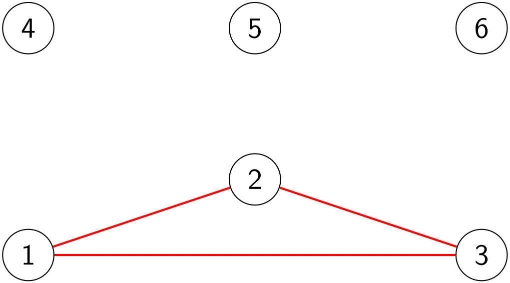
$\exists$ a mono triangle. We assume **R** and on vertices $\{1, 2, 3\}$.



We view $\{1, 2, 3\}$ and $\{4, 5, 6\}$ as the sides of a bipartite graph.
$\deg_{\mathbf{R}}(1)$ will mean the number of **R** edges between 1 and $\{4, 5, 6\}$.
$\deg_{\mathbf{R}}(4)$ will mean the number of **R** edges between 4 and $\{1, 2, 3\}$.

First Step

$\exists$ a mono triangle. We assume **R** and on vertices $\{1, 2, 3\}$.



We view $\{1, 2, 3\}$ and $\{4, 5, 6\}$ as the sides of a bipartite graph.
$\deg_{\mathbf{R}}(1)$ will mean the number of **R** edges between 1 and $\{4, 5, 6\}$.
$\deg_{\mathbf{R}}(4)$ will mean the number of **R** edges between 4 and $\{1, 2, 3\}$.
Generalize to $\deg_{\mathbf{R}}(v)$ and $\deg_{\mathbf{B}}(v)$.

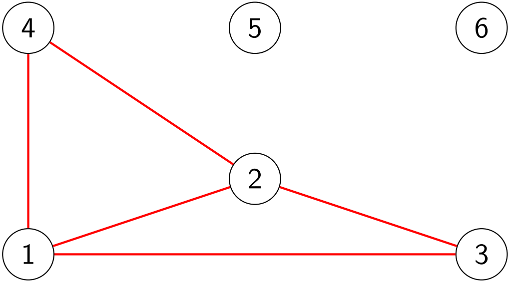
R Degree of $4, 5, 6$

$\mathbf{R}$ Degree of $4, 5, 6$

If $\exists v \in \{4, 5, 6\}$, $\deg_{\mathbf{R}}(v) \geq 2$ then get C_4:

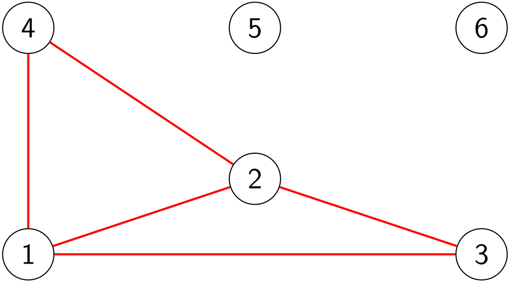
R Degree of $4, 5, 6$

If $\exists v \in \{4, 5, 6\}$, $\deg_{\mathbf{R}}(v) \geq 2$ then get C_4:



R Degree of $4, 5, 6$

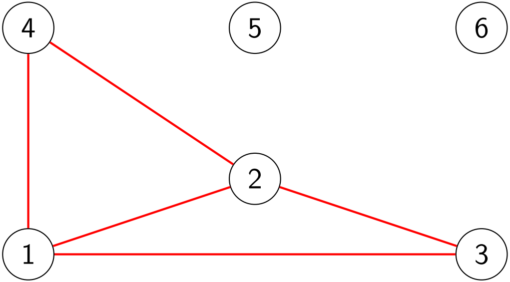
If $\exists v \in \{4, 5, 6\}$, $\deg_{\mathbf{R}}(v) \geq 2$ then get C_4:



C_4: $4 - 1 - 3 - 2 - 4$.

R Degree of $4, 5, 6$

If $\exists v \in \{4,5,6\}$, $\deg_{\mathbf{R}}(v) \geq 2$ then get C_4:

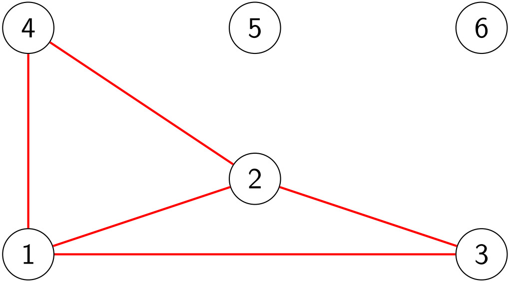


C_4: $4 - 1 - 3 - 2 - 4$.

Note: $(\forall v \in \{4,5,6\})[\deg_{\mathbf{B}}(v) \geq 2]$.

R Degree of $4, 5, 6$

If $\exists v \in \{4,5,6\}$, $\deg_{\mathbf{R}}(v) \ge 2$ then get C_4:



C_4: $4 - 1 - 3 - 2 - 4$.

Note: $(\forall v \in \{4,5,6\})[\deg_{\mathbf{B}}(v) \ge 2]$.
We will show that for $(\forall v \in \{4,5,6\})[\deg_{\mathbf{B}}(v) = 2]$.

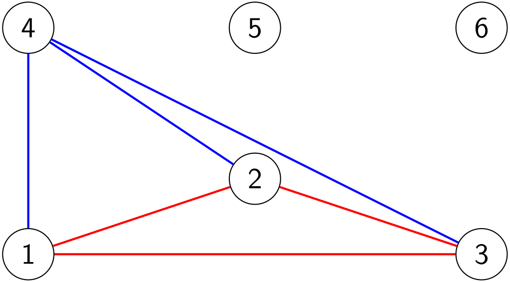
B Degree of $4, 5, 6$

B Degree of $4, 5, 6$

If $(\exists v \in \{4, 5, 6\})[\deg_{\mathbf{B}}(v) = 3]$ then get C_4:

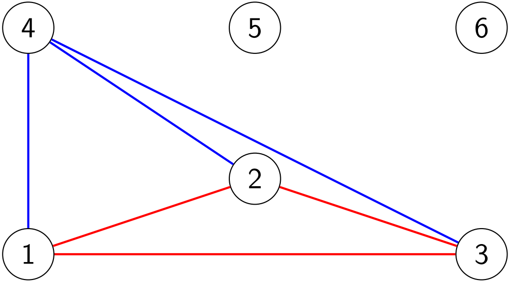
B Degree of $4, 5, 6$

If $(\exists v \in \{4,5,6\})[\deg_{\mathbf{B}}(v) = 3]$ then get C_4:



$\mathbf{B}$ Degree of $4, 5, 6$

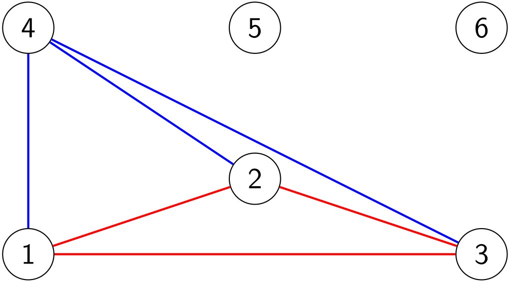
If $(\exists v \in \{4,5,6\})[\deg_{\mathbf{B}}(v) = 3]$ then get C_4:



Recall that $\deg_{\mathbf{B}}(5) \geq 2$.

B Degree of $4, 5, 6$

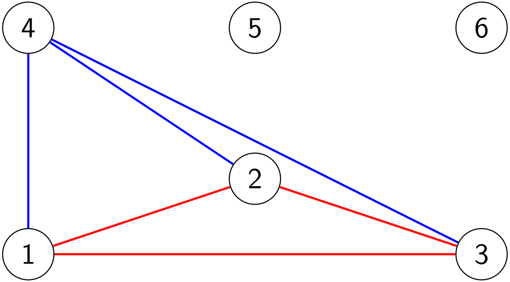
If $(\exists v \in \{4,5,6\})[\deg_{\mathbf{B}}(v) = 3]$ then get C_4:



Recall that $\deg_{\mathbf{B}}(5) \ge 2$.
If $\mathrm{COL}(5,1) = \mathrm{COL}(5,2) = \mathbf{B}$ then C_4: $5 - 1 - 4 - 2 - 5$.

B Degree of $4, 5, 6$

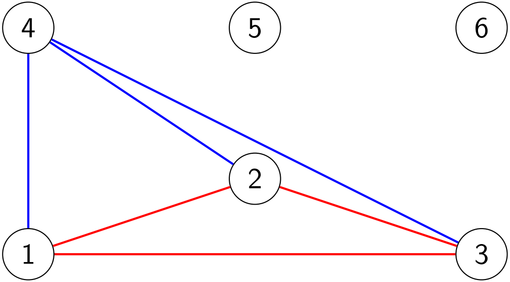
If $(\exists v \in \{4,5,6\})[\deg_{\mathbf{B}}(v) = 3]$ then get C_4:



Recall that $\deg_{\mathbf{B}}(5) \geq 2$.
If $\mathrm{COL}(5,1) = \mathrm{COL}(5,2) = \mathbf{B}$ then C_4: $5 - 1 - 4 - 2 - 5$.
If $\mathrm{COL}(5,2) = \mathrm{COL}(5,3) = \mathbf{B}$ then C_4: $5 - 3 - 4 - 2 - 5$.

B Degree of $4, 5, 6$

If $(\exists v \in \{4,5,6\})[\deg_{\mathbf{B}}(v) = 3]$ then get C_4:



Recall that $\deg_{\mathbf{B}}(5) \ge 2$.

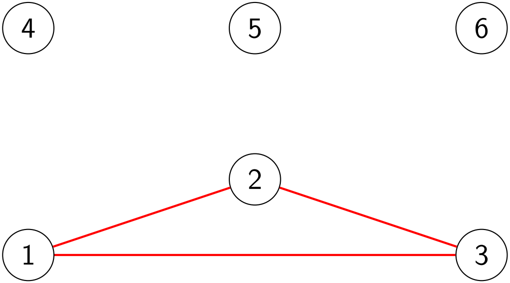
If $\mathrm{COL}(5,1) = \mathrm{COL}(5,2) = \mathbf{B}$ then C_4: $5 - 1 - 4 - 2 - 5$.

If $\mathrm{COL}(5,2) = \mathrm{COL}(5,3) = \mathbf{B}$ then C_4: $5 - 3 - 4 - 2 - 5$.

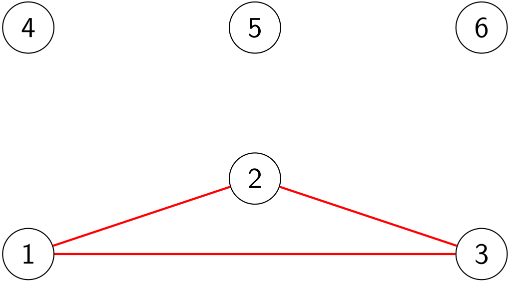
If $\mathrm{COL}(5,1) = \mathrm{COL}(5,3) = \mathbf{B}$ then C_4: $5 - 1 - 4 - 3 - 5$.

Recap

Recap

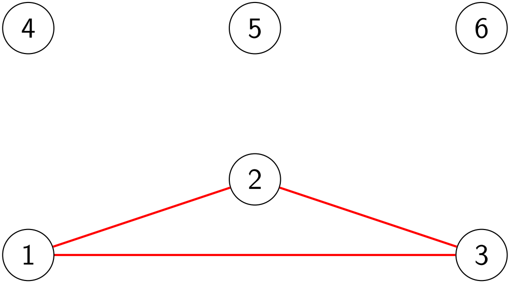


Recap



For $v \in \{4, 5, 6\}$, $\deg_{\mathbf{R}}(v) = 1$ and $\deg_{\mathbf{B}}(v) = 2$.

Recap



For $v \in \{4, 5, 6\}$, $\deg_{\mathbf{R}}(v) = 1$ and $\deg_{\mathbf{B}}(v) = 2$.

We now look at the degrees of $1, 2, 3$.

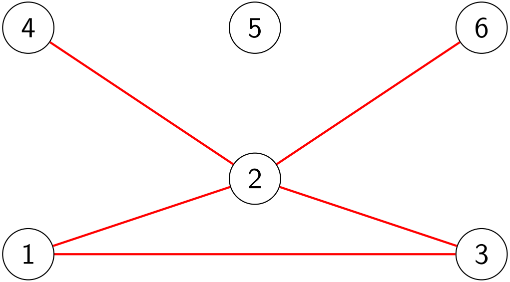
For $v \in \{1, 2, 3\}$, $\deg_{\mathbf{R}}(v) = 1$

For $v \in \{1, 2, 3\}$, $\deg_{\mathbf{R}}(v) = 1$

If $\exists v \in \{1, 2, 3\}$ $\deg_{\mathbf{R}}(v) \geq 2$ then $\mathbf{C_4}$ or $\mathbf{C_4}$.

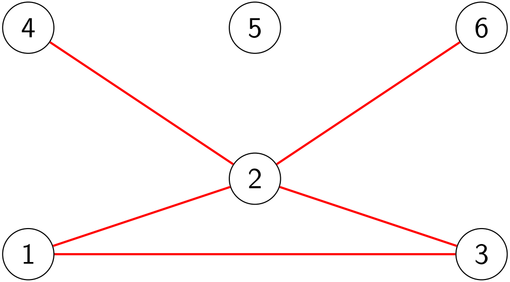
For $v \in \{1, 2, 3\}$, $\deg_{\mathbf{R}}(v) = 1$

If $\exists v \in \{1, 2, 3\}\ \deg_{\mathbf{R}}(v) \geq 2$ then $\mathbf{C_4}$ or $\mathbf{C_4}$.



For $v \in \{1,2,3\}$, $\deg_{\mathbf{R}}(v) = 1$

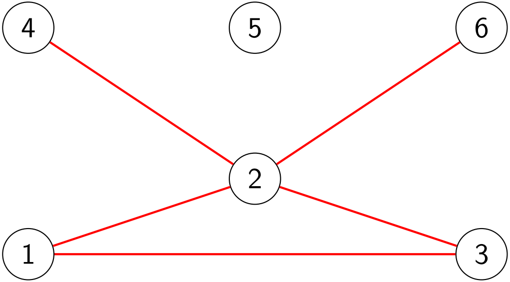
If $\exists v \in \{1,2,3\}$ $\deg_{\mathbf{R}}(v) \geq 2$ then $\mathbf{C_4}$ or $\mathbf{C_4}$.



If $\mathrm{COL}(1,4) =$ **R** then C_4: $1 - 4 - 2 - 3 - 1$.

For $v \in \{1,2,3\}$, $\deg_{\mathbf{R}}(v) = 1$

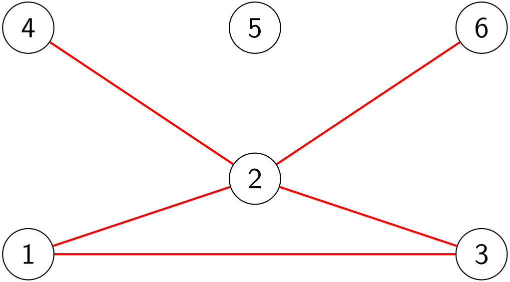
If $\exists v \in \{1,2,3\}\ \deg_{\mathbf{R}}(v) \geq 2$ then $\mathbf{C_4}$ or $\mathbf{C_4}$.



If $\mathrm{COL}(1,4) = \mathbf{R}$ then C_4: $1 - 4 - 2 - 3 - 1$.
If $\mathrm{COL}(3,6) = \mathbf{R}$ then C_4: $3 - 6 - 2 - 1 - 3$.

For $v \in \{1,2,3\}$, $\deg_{\mathbf{R}}(v) = 1$

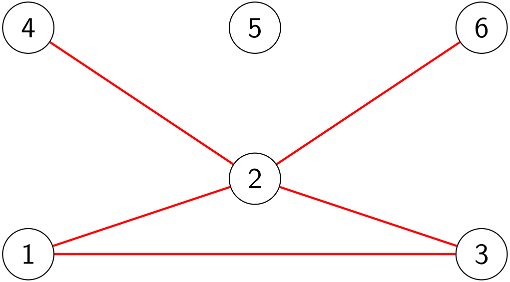
If $\exists v \in \{1,2,3\}$ $\deg_{\mathbf{R}}(v) \geq 2$ then $\mathbf{C_4}$ or $\mathbf{C_4}$.



If $\mathrm{COL}(1,4) = \mathbf{R}$ then C_4: $1-4-2-3-1$.
If $\mathrm{COL}(3,6) = \mathbf{R}$ then C_4: $3-6-2-1-3$.
If $\mathrm{COL}(3,4) = \mathbf{R}$ then C_4: $3-4-2-1-3$.

For $v \in \{1,2,3\}$, $\deg_{\mathbf{R}}(v) = 1$

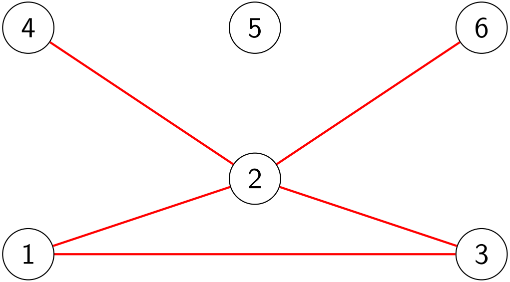
If $\exists v \in \{1,2,3\}$ $\deg_{\mathbf{R}}(v) \geq 2$ then $\mathbf{C_4}$ or $\mathbf{C_4}$.



If $\mathrm{COL}(1,4) = \mathbf{R}$ then C_4: $1 - 4 - 2 - 3 - 1$.
If $\mathrm{COL}(3,6) = \mathbf{R}$ then C_4: $3 - 6 - 2 - 1 - 3$.
If $\mathrm{COL}(3,4) = \mathbf{R}$ then C_4: $3 - 4 - 2 - 1 - 3$.
If $\mathrm{COL}(1,6) = \mathbf{R}$ then C_4: $1 - 6 - 2 - 3 - 1$.

For $v \in \{1,2,3\}$, $\deg_{\mathbf{R}}(v) = 1$

If $\exists v \in \{1,2,3\}$ $\deg_{\mathbf{R}}(v) \geq 2$ then $\mathbf{C_4}$ or $\mathbf{C_4}$.



If $\mathrm{COL}(1,4) = \mathbf{R}$ then C_4: $1 - 4 - 2 - 3 - 1$.
If $\mathrm{COL}(3,6) = \mathbf{R}$ then C_4: $3 - 6 - 2 - 1 - 3$.
If $\mathrm{COL}(3,4) = \mathbf{R}$ then C_4: $3 - 4 - 2 - 1 - 3$.
If $\mathrm{COL}(1,6) = \mathbf{R}$ then C_4: $1 - 6 - 2 - 3 - 1$.
If all of those edges are **B** then C_4: $1 - 4 - 3 - 6 - 1$.

Red Degree Recap

Red Degree Recap

We have

Red Degree Recap

We have

- $(\forall v \in \{1,2,3\}[\deg_{\mathbf{R}}(v) = 1]$.

Red Degree Recap

We have

- $(\forall v \in \{1,2,3\})[\deg_{\mathbf{R}}(v) = 1]$.
- $(\forall v \in \{4,5,6\})[\deg_{\mathbf{R}}(v) = 1]$.

Red Degree Recap

We have

- $(\forall v \in \{1,2,3\}[\deg_{\mathbf{R}}(v) = 1]$.
- $(\forall v \in \{4,5,6\}[\deg_{\mathbf{R}}(v) = 1]$.
- Hence we can assume

Red Degree Recap

We have

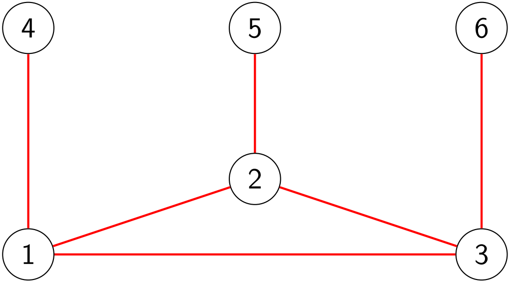
- $(\forall v \in \{1,2,3\}[\deg_{\mathbf{R}}(v) = 1]$.
- $(\forall v \in \{4,5,6\}[\deg_{\mathbf{R}}(v) = 1]$.
- Hence we can assume
  - $\mathrm{COL}(1,4) = \mathrm{COL}(2,5) = \mathrm{COL}(4,6) = \mathbf{R}$

Red Degree Recap

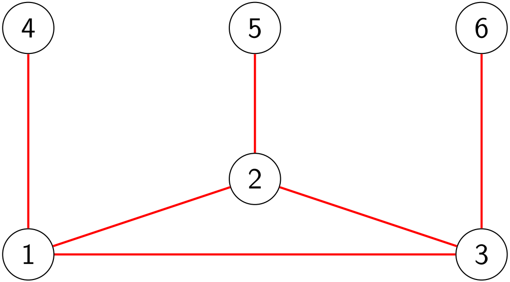
We have

- $(\forall v \in \{1,2,3\}[\deg_{\mathbf{R}}(v) = 1]$.
- $(\forall v \in \{4,5,6\}[\deg_{\mathbf{R}}(v) = 1]$.
- Hence we can assume
  - $\mathrm{COL}(1,4) = \mathrm{COL}(2,5) = \mathrm{COL}(4,6) = \mathbf{R}$
  - All other edges between $\{1,2,3\}$ and $\{4,5,6\}$ are **B**. (We will find some other edges that must be **B**.)

What We Know: R

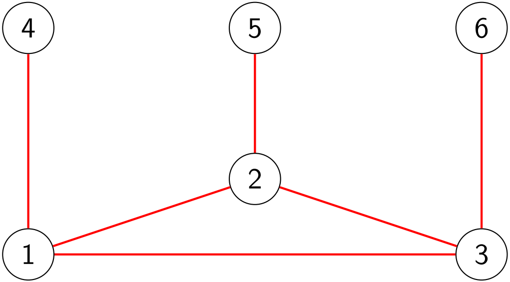


What We Know: R



All edges between $\{1, 2, 3\}$ and $\{4, 5, 6\}$ not shown are **B**.

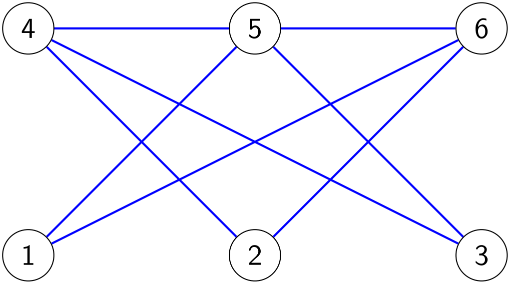
What We Know: R



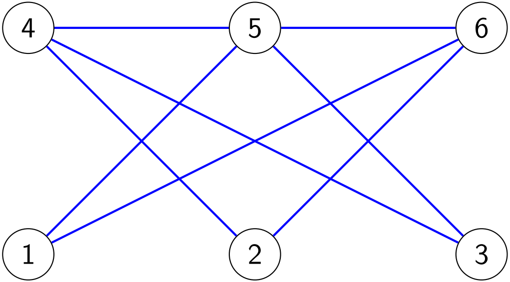
All edges between $\{1, 2, 3\}$ and $\{4, 5, 6\}$ not shown are **B**.

Clearly $\mathrm{COL}(4, 5) = \mathrm{COL}(5, 6) = $ **B**.

What We Know: B

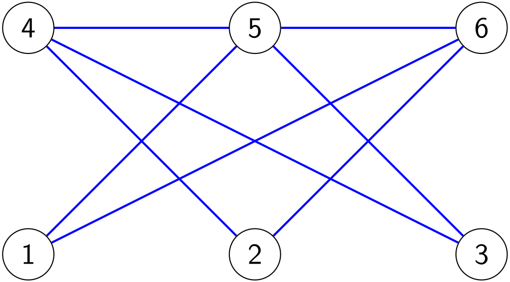


What We Know: B



C_4: $2 - 4 - 5 - 6 - 2$.

What We Know: B



C_4: 2 − 4 − 5 − 6 − 2.

DONE!