One Triangle, Two Triangles

William Gasarch

Lets Party Like Its 2019

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Thm For all 2-col of the edges of K_6 there is a mono K_3.

Trivial Theorem, Non Trivial Extension

Thm For all 2-cols of edges of K_{12} there are 2 mono K_3's
Question Find n such that

1. For all 2-col of the edges of K_n there are 2 mono K_3's
2. There exists a 2-col of the edges of K_{n-1} that does not have 2 mono K_3's.

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1. For all 2-col of the edges of K_6 there are **2** mono K_3's
2. There exists a 2-col of the edges of K_5 that does not have any mono K_3's.

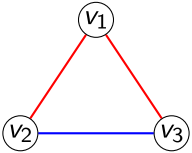
Proof of K_6 Two Triangles Theorem

Thm For all 2-cols of edges of K_6 there are 2 mono K_3's
Proof Let COL be a 2-col of the edges of K_6.
Let R, B, M, be the SET of **RED**, **BLUE**, and **MIXED** triangles.

$$|R| + |B| + |M| = \binom{6}{3} = 20.$$

We show that $|M| \leq 18$, so $|R| + |B| \geq 2$.

A Mixed Triangle Has a Vertex Such That



- (v_2, v_1) is red, (v_2, v_3) is blue. View this as $(v_2, \{v_1, v_3\})$.
- (v_3, v_1) is red, (v_3, v_2) is blue. View this as $(v_3, \{v_1, v_2\})$.

Map ZAN to M

Def A **Zan** is an element $(v, \{u, w\}) \in V \times \binom{V}{2}$ such that $v \notin \{u, w\}$ and $COL(v, u) \neq COL(v, w)$. ZAN is the set of Zan's.

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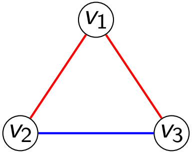

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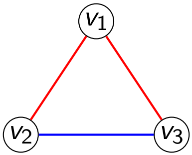

$(v_2, \{v_1, v_3\})$ and $(v_3, \{v_1, v_2\})$.

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So v contributes $\deg_R(v) \times \deg_B(v)$.

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$|ZAN| \leq 6 \times 6 = 36$.

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So there are at least 2 Mono Triangles.

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$$|M| = |ZAN|/2 \le \frac{(n-1)^2 n}{8}$$

Finishing Up The Proof

Recap

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Recall

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$$= \frac{n^3}{24} - \frac{n^2}{4} + \frac{5n}{24}$$

What About The Other Cases?

We leave the other cases to the reader to both determine the theorem and prove it.

Can This Be Improved?

The bound is known to be tight.

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1. As posed the answer is $n = 18$. Piwakoswki and Radziszowski https://www.cs.rit.edu/~spr/PUBL/paper40.pdf showed that for every 2-col of K_{18} there are 9 mono K_4's.

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2. I will present a math-interesting proof of the following: *For all 2-cols of K_{19} there are TWO mono K_4's.*

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Since $|A_i| = R(4)$, each A_i has a mono K_4.

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So we get $\binom{19}{18}$ mono K_4's. So we are almost there.

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For the real proof, see next slide.

Proof of K_{19} Two K_4 Theorem (cont)

List out all subsets of $V = \{1, \ldots, 19\}$ of size $R(4) = 18$.

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Those are our 2 mono K_4's.

Can the Proof Give 3 mono K_4's?

We show that the technique to get 2 mono K_4's cannot be extended to give 3 mono K_4's.

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Note We only showed that the **proof** cannot be extended. As noted above any 2-col of K_{18} has 9 mono K_4's.

Want n such that $\forall$ 2-col $\exists$3 Mono K_4's

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Want 3 Mono K_4's (cont)

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Thm $\forall$ 2-cols of the edges of K_{22} $\exists$ 3 mono K_4's.

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Since $\binom{n}{18} - m\binom{n-4}{14} \geq 1$ this process can go for $\geq m+1$ iterations and produce $\geq m+1$ mono K_4's.

Want m Mono K_4's (cont)

We just proved that for all $n, m \in \mathbb{N}$:
Thm If $\binom{n}{18} - m\binom{n-4}{14} \geq 1$ then $\forall$ 2-col of K_n $\exists$ $m+1$ mono K_4's.

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We state a theorem which expresses this in several ways, on the next slide.

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4) There are $\frac{n^4}{73440} - \frac{n^3}{12240} + \Omega(n^2)$ mono K_4's.

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There are $\frac{1}{3060}\binom{n}{4}$ mono K_4's.

Generalize

Left to the reader

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Left to the reader

1. Generalize to mono K_m.

Generalize

Left to the reader

1. Generalize to mono K_m.
2. Generalize to c colors.

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3. Generalize to c colors and mono K_m.