Combinatorial Proofs of Sane Bounds on Some Polynomial van der Waerden Numbers

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Abstract

Let $p \in \mathbb{Z}[x]$ and $c \in \mathbb{N}$. Then there exists a W such that, for any c-coloring of $[W]$, there exists a and d such that a and $a + p(d)$ are the same color. This is a special case of the Polynomial Van Der Waerden theorem. The known proofs give insane bounds on W. In this paper we give sane bounds for some p and $c = 2, 3, 4$.

1 Introduction

We use the following standard notation and definitions.

Def 1.1 Let $\mathbb{Z}$ be the set of integers, $\mathbb{N}$ be the set of non-negative integers, and $\mathbb{N}^+$ be the set of positive integers. If $W \in \mathbb{N}^+$ then let $[W]$ be the set $\{1, \ldots, W\}$.

Recall van Der Waerden's Theorem [16] (see also [5], [7]), which says that, for any c-coloring of a large enough initial segment of the naturals, there will large monochromatic arithmetic sequences. Formally:

Theorem 1.2 *For any $k, c \in \mathbb{N}$, there exists $W = W(k, c)$, such that for any c-coloring of $[W]$, there exists $a, d \in \mathbb{N}$, $d \neq 0$, such that $a, a + d, \ldots, a + (k-1)d$ are all the same color.*

The original proof by van der Waerden was purely combinatorial and yielded bounds on W that were INSANE (called EEEEEEEEEENORMOUS by [5]). In particular, the proof used an ω^2 induction and $W(k, c)$ was bounded by a function that is not primitive recursive. Shelah [15] gave a purely combinatorial proof that yielded bounds that were HUGE, though not INSANE. In particular the bounds were primitive recursive. Gowers [4] gave a proof using non-combinatorial (and difficult) techniques that yielded bounds that were much smaller than Shelah's bounds, but still HUGE:

$$W(k, c) \leq 2^{2^{c^{2^{2^{k+9}}}}}$$

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We discuss a known generalization of van der Waerden's theorem. Recall that the conclusion of van der Waerden's theorem is that

$$a, a+d, a+2d, \ldots, a+(k-1)d \text{ are the same color.}$$

Can we replace $d, 2d, \ldots, (k-1)d$ by other functions of d? Yes. We can replace them with polynomials with coefficients in $\mathbb{Z}$ and no constant term. Here is the Polynomial van Der Waerden Theorem:

Theorem 1.3 *Let $p_1(x), \ldots, p_k(x) \in \mathbb{Z}[x]$ such that, for $1 \le i \le k$, $p_i(0) = 0$. Let $c \in \mathbb{N}$. Then there exists $W = W(p_1(x), \ldots, p_k(x); c)$ such that, for any c-coloring of $[W]$, there exists $a, d \in \mathbb{N}$, $d \ne 0$, such that $a, a+p_1(d), \ldots, a+p_k(d)$ are all the same color.*

For $k = 1$ and $p_1(x) = x^2$, this theorem was proven independently by Furstenberg [3] and Sárközy [13]. Bergelson and Leibman [1] proved the full result using ergodic methods. The proofs by Furstenberg and Bergelson-Leibman yielded no upper bounds on $W(p_1(x), \ldots, p_k(x); c)$ (Sárközy proof did as we will see later.) Walters [17] proved Theorem 1.3 using combinatorial techniques, which yielded bounds on W that were INSANE. In particular, the proof used an ω^ω induction and $W(p_1(x), \ldots, p_k(x); c)$ was bounded by a function that is not primitive recursive. Once again Shelah [14] gave a purely combinatorial proof that yielded bounds that were HUGE, though not INSANE. In particular the bounds are primitive recursive. Peluse [8] has the best known upper bounds for sets of polynomials of distinct degrees. Peluse and Prediville [9] have the best known upper bounds for $W(x^2, x^2+x; c)$. With some effort one can write down these bounds in many cases (similar to Gowers bound on van Der Warden Numbers).

We are interested in the case of $W(ax^2+bx; c)$ where $c = 2, 3, 4$. Furstenberg's proof showed that $W(x^2; c)$ exists; however, his proof gave no upper bounds. Sárközy's proof showed that $W(x^2; c) \le 2^{O(c^3)}$. Pintz, Steiger, and Szemeredi [10] (see also [6] and [11] for exposition) showed that $W(x^2; c) \le 2^{O(c^{0.0001})}$ The 0.0001 can be replaced with any $\epsilon > 0$, however, in that case the constant associated with the big-O will increase. Similar comments apply below when we use 0.0001.

Harnel, Lyall, and Rice [6] showed that there exists a function $f: \mathbb{Z} \times \mathbb{Z} \to \mathbb{N}$ such that

$$W(ax^2+bx; c) \le 2^{f(a,b)c^{0.0001}}$$

Later Rice [11] showed that, for all k, there exists a function $f: \mathbb{Z}^k \to \mathbb{N}$ such that

$$W(a_k x^k + \cdots + a_1 x; c) \le 2^{f(a_k,\ldots,a_1)c^{0.0001}}$$

Rice [12] later obtained the following more precise result: for all $\epsilon > 0$, for all $a_1, \ldots, a_k \in \mathbb{Z}$, for $J = |a_1| + \cdots + |a_k|$:

$$W(a_k x^k + \cdots + a_1 x; c) \le 2^{2^{2^{2^{100k^2/\epsilon}}}} + 2^{2^{2^{(100k^4 \log J)^{100}}}} + 2^{c^\epsilon}$$

In summary, the known bounds on $W(ax^2 + bx; c)$ are HUGE.

In this paper we show that, for some $p(x) \in \mathbb{Z}[x]$ and $c = 2, 3, 4$, one can obtain sane bounds on $W(p(x); c)$. Our proofs will be purely combinatorial and easier than those Walters, Shelah, Peluse, Prediville, and Rice. We hasten to point out that they proved far more general cases of the poly van der Warden theorem whereas we only prove it in special cases.

We show the following.

- For all $a \in \mathbb{Z}$, $W(ax; c) = |ac| + 1$.
- $W(x^n; 2) = 2^n + 1$ and, for all $a \in \mathbb{Z}$, $W(ax^n; 2) = 2^n a + 1$.
- Let $p(x) \in \mathbb{Z}[x]$ such that $p(0) = 0$. Then $W(p(x); 2)$ is bounded above by the min of $|p(i)| + |p(j)| - g + 1$ such that (a) $i, j \in \mathbb{N}$, (b) $p(i), p(j) \neq 0$, (c) $g = \gcd(p(i), p(j))$, (d) either $p(i)/g$ or $p(j)/g$ is even. Appendix A has a table of some exact values of $W(ax^2 + bx; 2)$.
- $W(x^2; 3) = 29$ and, for all $a \in \mathbb{Z}$, $W(ax^2; 3) = 28a + 1$. Appendix B has a table of some exact values of $W(ax^2 + bx; 3)$.
- $W(x^2; 4) \le$ 84,149,474,894,213,522. Appendices C, D, and E have tables of some upper bounds on $W(ax^2 + bx; 4)$.

2 Preliminaries

We are concerned with colorings of initial segments of $\mathbb{N}$ that avoid certain distances between same-colored naturals. For example an $(x^2; 4)$-proper coloring of $[1000]$ would be a 4-coloring of $[1000]$ where no points that are a square apart are the same color. More generally, we have the following definition.

Def 2.1 Let $c \in \mathbb{N}^+$ and $W \in \mathbb{N}^+$.

1. A *c-coloring of* $[W]$ is a mapping $[W] \to [c]$.
2. Let $p(x) \in \mathbb{Z}[x]$. A $(p(x); c)$*-proper coloring of* $[W]$ is a c-coloring of $[W]$ such that, for all $x, y \in [W]$, if $y - x = p(d)$ for some $d \in N^+$, then x and y have different colors. When the context is clear, we will often write *proper c-coloring* or simply *proper coloring*.

Note that the polynomial van der Waerden number, $W = W(p(x); c)$, is the least number such that there is no $(p(x); c)$-proper coloring of $[W]$. Although we care about proper $(p(x); c)$-colorings, we need a more general notion:

Def 2.2 Let $F \subseteq \mathbb{Z}$, $c \in \mathbb{N}^+$, and $W \in \mathbb{N}^+$.

- An $(F; c)$*-proper coloring of* $[W]$ is a c-coloring of $[W]$ such that, for all $x, y \in [W]$ with $y - x \in F$, x and y have different colors.
- $W = W(F; c)$ is the least number such that there is no $(F; c)$-proper coloring of $[W]$. If no such number exists, we set $W(F; c) = \infty$.

We leave the following easy lemma to the reader.

Lemma 2.3 *Let* $c \in \mathbb{N}^+$*.*

1. *If* $0 \in F$ *then* $W(F; c) = 1$*.*
2. *Assume* $f \in F$*. Let* $F' = F \cup \{-f\}$*. Then* $W(F; c) = W(F'; c)$*.*

We prove an easy theorem. The techniques to prove it yield a lemma that we will use later.

Theorem 2.4

1. $W(x^2; 2) = 5 = 4 + 1$*.*
2. $W(ax^2; 2) = (W(x^2; 2) - 1)a + 1 = 4a + 1$*.*

Proof:
1) $W(x^2; 2) \leq 5$: Assume, by way of contradiction, that COL is an $(x^2; 2)$-proper coloring of $[5]$. We can assume COL(1) $= R$. Since 1 is a square we have COL(2) $= B$, COL(3) $= R$, COL(4) $= B$, COL(5) $= R$. Then COL(1) $=$ COL(5) with $5 - 1 = 2^2$, which is a contradiction.

$W(x^2; 5) \geq 5$ via the following $(x^2; 2)$-proper coloring of $[4]$:

1	2	3	4
R	B	R	B

2) $W(ax^2; 2) \leq (W(x^2; 2) - 1)a + 1 = 4a + 1$: Assume, by way of contradiction, that COL is an $(ax^2; 2)$-proper coloring of $[4a + 1]$. We use COL to define COL$'$, an $(x^2; 2)$-proper coloring of $[5]$.

$\mathrm{COL}'(1) = \mathrm{COL}(1)$
$\mathrm{COL}'(2) = \mathrm{COL}(a+1)$
$\mathrm{COL}'(3) = \mathrm{COL}(2a+1)$
$\mathrm{COL}'(4) = \mathrm{COL}(3a+1)$
$\mathrm{COL}'(5) = \mathrm{COL}(4a+1)$

By using the fact that a and $4a$ are forbidden distances for COL, one can show that COL′ is an $(x^2; 2)$-proper coloring of [5], which is a contradiction.

$W(ax^2; 2) \geq (W(x^2; 2) - 1)a + 1 = 4a + 1$: Let COL be an $(x^2; 2)$-proper coloring of [4]. We use COL to define COL′, an $(ax^2; 2)$-proper coloring of $[4a]$.

Let $1 \leq x \leq 4a$. Let $0 \leq i \leq 3$, and $1 \leq j \leq a$ be such that $x = ia + j$. Let

$$\mathrm{COL}'(x) = \mathrm{COL}'(ia+j) = \mathrm{COL}(i+1).$$

By using that COL is an $(x^2; 2)$-proper coloring of [4], one can show that COL′ is an $(ax^2; 2)$-proper coloring of $[4a]$. ∎

Using the ideas behind Theorem 2.4.2 one can show the following:

Lemma 2.5 *Let $p(x) \in \mathbb{Z}[x]$, $a \in \mathbb{Z}$, and $c \in \mathbb{N}$. Then $W(ap(x); c) = a(W(p(x); c) - 1) + 1$.*

3 Linear polynomials

For completeness we cover linear polynomials, for which we obtain a complete solution.

Theorem 3.1 *Let $a \in \mathbb{Z}$ and $c \in \mathbb{N}^+$. Then*

$$W(ax; c) = |ac| + 1 \ .$$

Proof: The case where $a = 0$ follows from Lemma 2.3.1. For $a \neq o$ we have that $|a|$ is a forbidden distance.

$W(ax; c) \leq |ac| + 1$: By setting $x = 1, 2, \ldots, c$ we get forbidden distances $|a|, |2a|, \ldots, |ca|$. So $1, |a| + 1, |2a| + 1, \ldots, |ca| + 1$ must all be different colors, but there are only c colors.

$W(ax; c) \geq |ac| + 1$: We can properly c-color $[ca]$: color $1, \ldots, |a|$ by 1, color $|a|+1, \ldots, |2a|$ by 2, …, color $|(c-1)a| + 1, \ldots, |ca|$ by $c - 1$. ∎

4 Upper Bounds on $W(p(x); 2)$ for any $p(x) \in \mathbb{Z}[x]$

The following is our main lemma.

Lemma 4.1 *Let $s, t \in \mathbb{N}^+$. Let $g = \gcd(s,t)$. Then*

$$W(\{s,t\};2) = \begin{cases} s+t-g+1 & \textit{if either } s/g \textit{ or } t/g \textit{ is even} \\ \infty & \textit{otherwise.} \end{cases}$$

Proof:
Temporarily assume s and t are relatively prime, so $g = 1$.

Let $z = s + t$. Let COL be a $(\{(s,t\}; c)$-proper coloring of $[z-1]$. We are *not* aiming for a contradiction; we are aiming to see that the entire coloring is forced.

Consider the list

$$s \bmod z,\ 2s \bmod z,\ 3s \bmod z,\ \ldots,\ (z-1)s \bmod z.$$

The absolute value of the difference of every pair of adjacent values is s or t. Hence $2s \bmod z$ is B, $3s \bmod z$ is R, $4s \bmod z$ is B, etc.

Since s is relatively prime to t, it is also relatively prime to z. Hence

$$\{s \bmod z,\ 2s \bmod z,\ \ldots,\ (z-1)s \bmod z\} = [z-1].$$

Therefore we have forced a 2-coloring of (all of) $[z-1]$. We discuss if the coloring can be extended beyond $z-1$.

Extend Beyond $z-1$?: Whether this proper coloring can be extended beyond $[z-1]$ depends on the parity of z:

CASE (1): Assume that either s or t is even. (The other must be odd because we have assumed that $g = 1$.)

Then $z - 1 = s + t - 1$ must be even, so that the first number in the above alternating list of colors, $s \bmod z$, and the last number, $(z-1)s \bmod z$, must have different colors. But

$$(z-1)s \equiv zs - s \equiv -s \equiv t \pmod{z}.$$

So $z \equiv s + t$ cannot be R or B, implying that the coloring cannot be extended to z.

CASE (2): Assume s and t are both odd.

The above alternating list of colors makes the odd numbers all have the same color, say R, and the even numbers B (because each addition changes the parity of the number being colored). Any number at or above $s+t$ can be colored, but its color is forced by subtracting s (or equivalently t). So the coloring can be uniquely extended to ∞.

We have proven the theorem in the case of $g = 1$. If $g \geq 2$ then there is no interaction of numbers x, y where $x \not\equiv y \pmod{g}$. We leave it to the reader to use this to prove the $g \geq 2$

case. ▮

Theorem 4.2 *Let $p(x) \in \mathbb{Z}[x]$ be a polynomial such that $p(0) = 0$. For $i, j \in \mathbb{N}$ let $g_{i,j} = \gcd(p(i), p(j))$. Then $W(p(x); 2)$ is bounded above by the min of $\{|p(i)| + |p(j)| - g_{i,j} + 1\}$ such that*

- $i, j \in \mathbb{N}$
- $p(i), p(j) \neq 0$
- *Either $p(i)/g_{i,j}$ or $p(j)/g_{i,j}$ is even.*

Proof:
Follows from Lemma 4.1. ▮

Corollary 4.3 *Let $n \geq 1$.*

1. $W(x^n; 2) = 2^n + 1$.
2. $W(ax^n; 2) = (W(x^n; 2) - 1)a + 1 = a2^n + 1$. *(This follows from Part 1 and Lemma 2.5.)*

Proof:
$W(x^n; 2) \leq 2^n + 1$.

Let $p(x) = x^n$. Let $i = 1$, $j = 2$, and $g = \gcd(1^n, 2^n) = 1$. Since (1) $i, j \in \mathbb{N}$ and (2) $p(j)/g = 2^n/1 = 2^n$ is even (since $n \geq 1$), $p(x), i, j$ satisfy the conditions of Theorem 4.2. Hence

$$W(x^n; 2) \leq 1^n + 2^n - g + 1 = 2^n + 1.$$

$W(x^n; 2) \geq 2^n + 1$. We present a proper 2-coloring of $[2^n]$. Color the even numbers R and the odd numbers B. Since $2^n - 1 < 2^n$, the only forbidden distance is 1. Hence this is a proper coloring. ▮

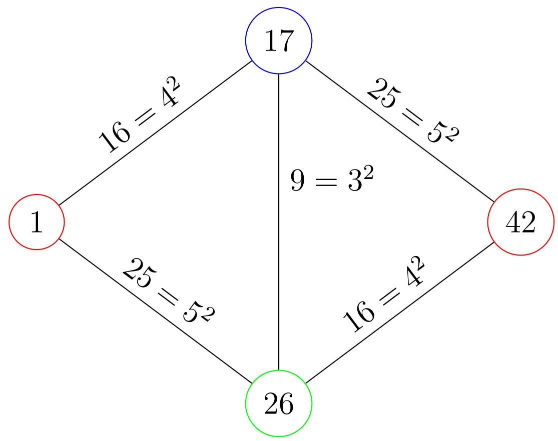

Figure 1: In any $(x^2;3)$-proper coloring, $\mathrm{COL}(x) = \mathrm{COL}(x+41)$

5 $W(ax^2;3) = 28a+1$

In this section we show that $W(x^2;3) = 29$ and then $W(ax^2;3) \le 28a+1$. We first show a weaker theorem which will be a good warm-up to our work on 4-colorings in Section 7.

Theorem 5.1 $W(x^2;3) \le 1 + 41^2 = 1682$.

Proof:
Assume, by way of contradiction, that COL is an $(x^2;3)$-proper coloring of $[1+41^2]$. We can assume $\mathrm{COL}(1) = R$ and $\mathrm{COL}(17) = B$. By Figure 1 we know that $\mathrm{COL}(26) \notin \{R, B\}$, hence $\mathrm{COL}(26) = G$. Again, by Figure 1, we have that $\mathrm{COL}(42) \notin \{B, G\}$, hence $\mathrm{COL}(42) = R$.

Note that we have shown that $\mathrm{COL}(1) = \mathrm{COL}(42)$. More generally we have shown that, for all x, $\mathrm{COL}(x) = \mathrm{COL}(x+41)$. Hence

$$\mathrm{COL}(1) = \mathrm{COL}(1+41) = \mathrm{COL}(1+2\times 41) = \cdots = \mathrm{COL}(1+40\times 41) = \mathrm{COL}(1+41^2).$$

This contradicts COL being an $(x^2;3)$-proper coloring. ∎

The following theorem was proven independendly by Matt Jordan.

Theorem 5.2

1. $W(x^2;3) = 29$.

2. For all $a \in \mathbb{Z}$, $W(ax^2;3) = 28a + 1$. This follows from Part 1 and Lemma 2.5.

Proof:

$W(x^2;3) \le 29$: Assume, by way of contradiction, that there exists COL, a proper 3-coloring of [29].

By Figure 2, $\mathrm{COL}(10) = \mathrm{COL}(17)$. By similar reasoning one can show that

$$(\forall x)[10 \le x \le 13 \implies \mathrm{COL}(x) = \mathrm{COL}(x+7)].$$

We refer to this fact as FORCE.

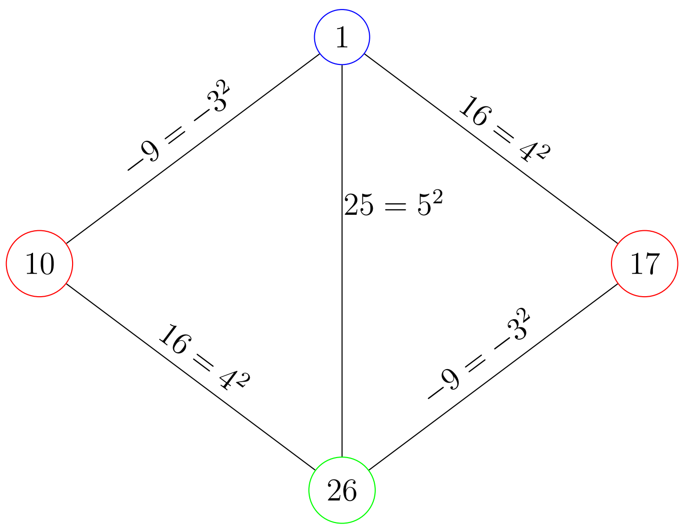

Figure 2: In any proper $(x^2;3)$-coloring, $\mathrm{COL}(10) = \mathrm{COL}(17)$

We can assume, without loss of generality, that $\mathrm{COL}(10) = R$. Since $11 - 10 = 1^2$ we know that $\mathrm{COL}(11) \ne R$. We can assume, without loss of generality, that $\mathrm{COL}(11) = B$.

17: By FORCE, $\mathrm{COL}(17) = \mathrm{COL}(10) = R$

18: By FORCE, $\mathrm{COL}(18) = \mathrm{COL}(11) = B$.

10	11	12	13	14	15	16	17	18	19	20
R	B						R	B		

19: Since $\mathrm{COL}(10) = R$ and $\mathrm{COL}(18) = B$, $\mathrm{COL}(19) = G$.

12: By FORCE, $\mathrm{COL}(12) = \mathrm{COL}(19) = G$.

10	11	12	13	14	15	16	17	18	19	20
R	B	G					R	B	G	

20: Since COL(11) = B and COL(19) = G, COL(20) = R.

13: By FORCE, COL(13) = COL(20) = R.

10	11	12	13	14	15	16	17	18	19	20
R	B	G	R				R	B	G	R

Now we have that COL(17) = COL(13) = R. But $17 - 13 = 2^2$. This is a contradiction.

$W(x^2, 3) \geq 29$:

We present a proper 3-coloring of [28]:

1	2	3	4	5	6	7	8	9	10	11	12	13	14
B	G	R	G	R	B	B	B	G	R	B	G	B	G

15	16	17	18	19	20	21	22	23	24	25	26	27	28
R	B	R	B	G	R	B	R	B	G	R	G	R	B

∎

Note 5.3 By Figure 2 we easily show $W(x^2; 3) \leq 68$: For $10 \leq x \leq 52$ COL(x) = COL($x+7$), so

$$\mathrm{COL}(10) = \mathrm{COL}(17) = \cdots = \mathrm{COL}(59),$$

and note that $59 - 10 = 49 = 7^2$. This result is not as strong as $W(x^2; 3) \leq 29$; however, it is a simpler proof and gives a better bound then the 1682 of Theorem 5.1.

6 Upper Bounds on $W(ax^2 + bx; 3)$

Def 6.1

(a) A coloring of $[n]$ has *repeat distance* r if x and $x + r$ have the same color, for all $1 \leq x \leq n - r$.

(b) A coloring of $[n]$ has *repeat distance* r *under one-sided boundary condition* b if x and $x + r$ have the same color, for all $1 \leq x \leq n - r - b$.

(c) A coloring of $[n]$ has *repeat distance* r *under two-sided boundary condition* b if x and $x + r$ have the same color, for all $b < x \leq n - r - b$.

Lemma 6.2 *In any proper 3-coloring of* $[n]$ *with forbidden distances* $s, t, s+t$*, where* $0 < s < t$*:*

(a) $2s + t$ *is a repeat distance.*

(b) $t - s$ *is a repeat distance under two-sided boundary condition* s*.*

(c) $3s$ *is a repeat distance under one-sided boundary condition* t*.*

Proof: Let $u = s + t$.

(a) Consider a 3-coloring satisfying the conditions of the lemma. Let $1 \le x \le n-(2s+t)$. Without loss of generality, we can assume that x is R. Then $x+s$ is not R, say B, and $x+u = (x+s)+t$ cannot be R or B so it must be G. Then $(x+s)+u = (x+u)+s$ cannot be B or G so it must be R. Since x and $x+u+s$ are both R, $(x+u+s)-x = u+s = 2s+t$ is a repeat distance,

(b) Consider a 3-coloring satisfying the conditions of the lemma. Let $s < x \le n-(t-s)-s$. Without loss of generality, we can assume that x is R. Then $x-s$ is not R, say B, and $(x-s)+u = x+t$ cannot be R or B so it must be G. Then $(x-s)+t = (x+t)-s$ cannot be B or G, so it must be R. This process requires that $x-s > 0$ and $x+t \le n$. So $(x+t-s)-x = t-s$ is a repeat distance under two-sided boundary condition s.

(c) Take $2s+t$ from Part (a) and subtract $t-s$ from Part (b). The repeat distance is $(2s+t)-(t-s) = 3s$. There is a one-sided boundary of size $(t-s)+s = t$ from one side of Part (b).

∎

Lemma 6.3 *Assume* $[w]$ *has a proper 3-coloring where* s *is a forbidden distance and* r *is a repeat distance under two-sided boundary condition* b*. If* $r|s$ *then*

$$w \le s + 2b + 1 \ .$$

Proof: Assume $w > s+2b+1$. Assume, without loss of generality, that $b+1$ is R. Then, since r is a repeat distance, $r+b+1$, $2r+b+1$, ..., $s+b+1$ are also R, since $b+1 > b$ and $(s+b+1)+b = s+2b+1 \le n$. But s is a forbidden distance so $b+1$ and $s+b+1$ cannot both be R. Contradiction. ∎

We give an example of using Lemma 6.3 to get an upper bound on a set of poly Van der Warden numbers. For one of them we have an exact value.

Theorem 6.4

1. *For $a, b > 0$ and $a|b$, $W(ax^2 + bx; 3) \leq 72b^2/a + 1$.*
2. *$W(x^2 + x; 3) = 73$.*

Proof: 1) Let $p(x) = ax^2 + bx$. Let

$$x = 5b/a, \qquad y = 6b/a, \qquad z = 8b/a \ .$$

Then

$$p(x) = 30b^2/a, \qquad p(y) = 42b^2/a, \qquad p(z) = 72b^2/a \ .$$

Since $p(x) + p(y) = p(z)$, by Lemma 6.2b, $p(y) - p(x) = 12b^2/a$ is a repeat distance under two-sided boundary condition $30b^2/a$. But $p(3b/a) = 12b^2/a$ is a forbidden distance. Thus, by Lemma 6.3, $W(ax^2 + bx; 3) \leq 12b^2/a + 2 \cdot 30b^2/a + 1 = 72b^2/a + 1$.

2) By Part 1 $W(x^2 + x; 3) \leq 73$. We show $W(x^2 + x; 3) \geq 73$ by giving a $(x^2 + x; 3)$-proper coloring of [72].

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
R	R	G	G	R	R	B	B	R	R	B	B	G	G	B	B	G	G

19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36
R	R	G	G	R	R	B	B	R	R	B	B	G	G	B	B	G	G

37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54
R	R	G	G	R	R	B	B	R	R	B	B	G	G	B	B	G	G

55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70	71	72
R	R	G	G	R	R	B	B	R	R	B	B	G	G	B	B	G	G

∎

7 Upper Bounds on $W(x^2; 4)$

Recall that Figure 1 was the key to showing $W(x^2; 3) \leq 1682$. We now derive parameters for a new figure that will be the key to an upper bound on $W(x^2; 4)$.

We need to find $a, b, c, d, e, f, x, y, z \in \mathbb{N}^+$ such that the following figure can be drawn:

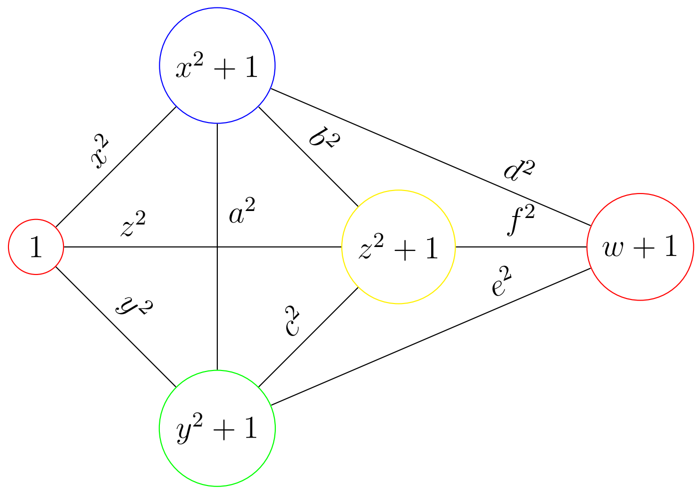

Figure 3: In any $(x^2; 4)$-proper coloring, COL(1) = COL(1 + w)

Hence we need to find solutions in $\mathbb{N}^+$ to the following system of equations:

$$\begin{aligned} x^2 + a^2 &= y^2 \\ x^2 + b^2 &= z^2 \\ y^2 + c^2 &= z^2 \\ x^2 + d^2 &= w \\ y^2 + e^2 &= w \\ z^2 + f^2 &= w \end{aligned}$$

Each equation is a Pythagorean triple, for which we have a known formula with parameters k, m, n where $m > n$, and m, n are coprime but not both odd; we can use the Farey sequence as an efficient algorithm to generate coprime pairs m, n. We used a computer program and obtained the following:

Theorem 7.1 $PW(4, x^2) \leq 1 + 290{,}085{,}289^2 = 84{,}149{,}474{,}894{,}213{,}522$

Proof:

Assume, by way of contradiction, that COL is an $(x^2; 4)$-proper coloring of $[1{+}290{,}085{,}289^2]$ By Figure 4 we know that

$$\mathrm{COL}(1) = \mathrm{COL}(1 + 290{,}085{,}289^2).$$

More generally we have shown that, for all x,

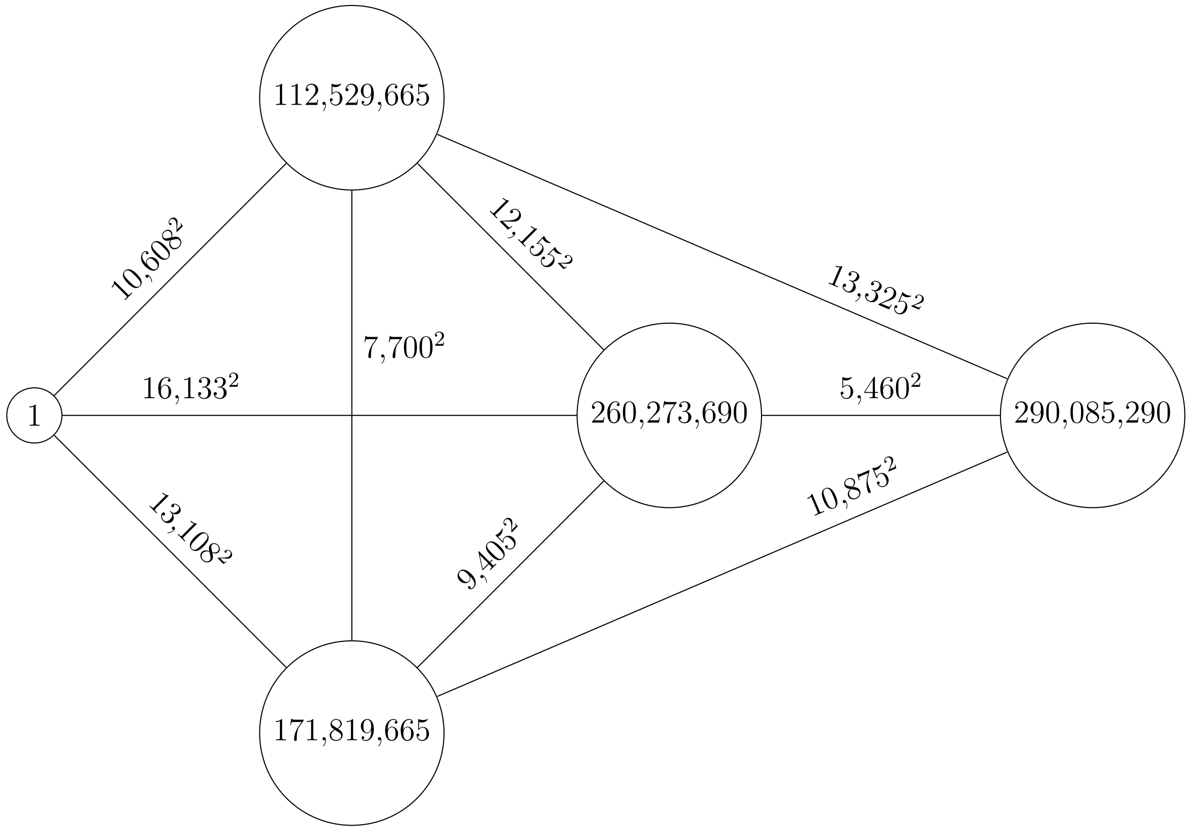

Figure 4: In any $(x^2; 4)$-proper coloring, $\mathrm{COL}(1) = \mathrm{COL}(1 + 290, 085, 290)$

$$\mathrm{COL}(x) = \mathrm{COL}(x + 290{,}085{,}289^2).$$

Hence

$$\mathrm{COL}(1) = \mathrm{COL}(1 + 1 \times 290{,}085{,}289) = \cdots = \mathrm{COL}(1 + 290{,}085{,}289^2).$$

This contradicts COL being an $(x^2; 4)$-proper coloring. ∎

Note 7.2 The program used to find these bounds, and an explanation of that program, are at `https://github.com/zaprice/polyvdw`

The actual value of $W(x^2; 4)$ is known to be 58 by the use of a SAT solver [2]. The bound in Theorem 7.1 is of course much worse, but the proof is very easy to verify.

8 Open Problems

The over-arching open problem is to get better bounds for $W(p(x);c)$ for a variety of $p(x) \in \mathbb{Z}[x]$ and $c \in \mathbb{N}^+$ using elementary methods (and not using a SAT Solver). In particular we would like to see the following proven that way:

1. A better bound on $W(x^2;4)$.

2. A bound on $W(ax^2+bx;4)$ that is a polynomial in $|a|$ and $|b|$.

3. A reasonable bound on $W(a_kx^k+\cdots+a_1;c)$.

A Some Exact Values of $W(ax^2+bx;2)$

Chart of $W(p(x);2)$ for $p(x)=ax^2+bx$ for $0 \le a \le 10$ and $-10 \le b \le 10$.
The values for $a,b \ge 0$ were obtained from our formulas.

		a										
		0	1	2	3	4	5	6	7	8	9	10
	-10	21	1	1	9	9	1	25	11	13	17	1
	-9	19	1	9	1	7	5	7	37	15	1	23
	-8	17	1	1	7	1	7	9	13	1	21	25
	-7	15	1	7	5	5	25	11	1	19	61	29
	-6	13	1	1	1	5	9	1	17	21	25	73
	-5	11	1	5	13	7	1	15	49	25	29	31
	-4	9	1	1	5	1	13	17	23	25	33	37
	-3	7	1	3	1	11	37	19	25	31	73	41
	-2	5	1	1	9	13	19	49	29	33	39	41
	-1	3	1	7	25	17	21	27	61	37	41	47
b	0	1	5	9	13	17	21	25	29	33	37	41
	1	3	13	13	17	23	49	33	37	43	85	53
	2	5	11	25	21	25	31	33	41	45	51	97
	3	7	13	19	37	29	33	37	73	49	49	59
	4	9	17	21	27	49	37	41	47	49	57	61
	5	11	25	25	29	35	61	45	49	55	97	61
	6	13	23	25	31	37	43	73	53	57	61	65
	7	15	25	31	49	41	45	51	85	61	65	71
	8	17	29	33	39	41	49	53	59	97	69	73
	9	19	37	37	37	47	73	55	61	67	109	77
	10	21	35	49	45	49	51	57	65	69	75	121

The numbers tend to increase with increasing a and $|b|$. When $a = b$ the values tend to be large; this is because neither $p(1)/g$ nor $p(2)/g$ is even so $W(p(x); 2) = p(3) + 1$, which is somewhat larger than $p(1) + p(2) - g + 1$ (the other possibility).

B Some Exact Values of $W(ax^2 + bx; 3)$

Chart of $W(p(x); 3)$ for $p(x) = ax^2 + bx$ for $0 \leq a \leq 5$ and $-5 \leq b \leq 5$.
The values were obtained by computer.

		a					
		0	1	2	3	4	5
	-5	16	1	64	61	217	1
	-4	13	1	1	91	1	289
	-3	10	1	10	1	135	171
	-2	7	1	1	68	97	171
	-1	4	1	49	105	190	183
b	0	1	29	57	85	113	141
	1	4	73	76	65	156	253
	2	7	64	145	123	151	?
	3	10	37	95	217	?	?
	4	13	65	127	?	289	?
	5	16	55	?	109	?	361

C Some Upper Bounds on $W(x^2 + Bx; 4)$

We give bounds for $W(x^2 + Bx; 4)$. Each row of he table gives B, x, y, z, w (as in Figure 5), and the bound. We have data for $0 \leq B \leq 2000$; however, we only present data for $0 \leq B \leq 31$ (we choose 31 since the 10 smallest values occur within $0 \leq B \leq 31$) and $1980 \leq B \leq 2000$ We choose 0 to 31 since the 10 smallest values occur within this range. We choose 1980 to 2000 since after 2000 we ran into computational issues. There is a tendency for larger B's to lead to larger bounds (with one outlier) In particular:

1. The smallest upper bound occurs when $B = 9$. The next smallest when $B = 3$. The ten smallest upper bounds occur when:
$$B = 9, 3, 23, 19, 25, 4, 5, 18, 6, 31.$$
2. The largest upper bound occurs when $B = 0$ (this is the outlier). The next largest when $B = 1309$. The ten largest upper bounds occur when:

$$B = 0, 1309, 743, 1787, 1171, 1727, 1386, 1847, 1993, 1877.$$

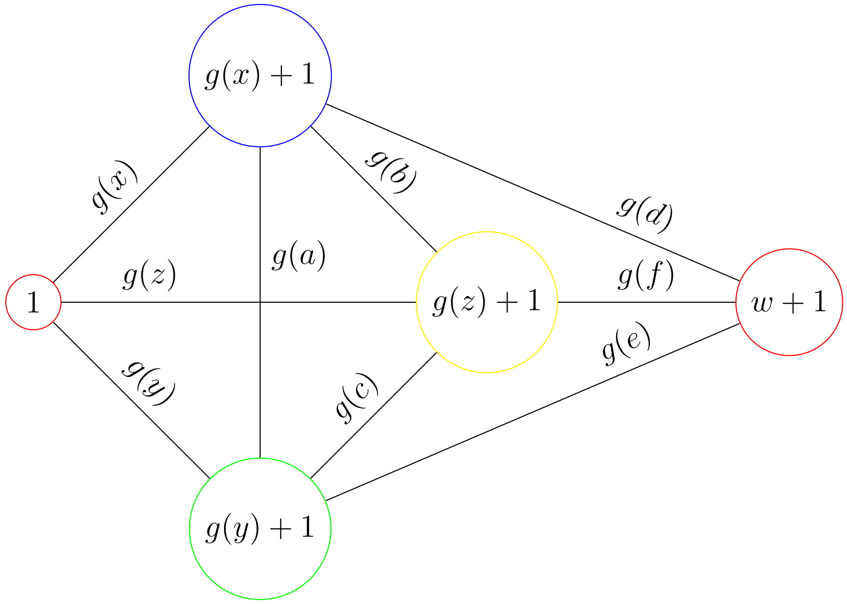

Figure 5: In any $(g(x); 4)$-proper coloring, $\mathrm{COL}(1) = \mathrm{COL}(1 + w)$

Table for $x^2 + Bx$ where $0 \leq B \leq 20$.

g	x	y	z	w	$W(g(x);4) \leq$
x^2	10,608	13,108	16,133	290,085,289	84,149,474,894,213,522 (largest)
x^2+x	299	302	327	113,262	12,828,393,907
x^2+2x	91	127	211	257,463	66,287,711,296
x^2+3x	35	43	53	3,308	10,952,789 (2nd smallest)
x^2+4x	80	84	92	10,197	104,019,598 (6th smallest)
x^2+5x	70	81	100	11,250	126,618,751 (7th smallest)
x^2+6x	70	86	106	13,232	175,165,217 (9th smallest)
x^2+7x	638	785	923	988,338	976,818,920,611
x^2+8x	160	168	184	40,788	1,663,987,249
x^2+9x	35	37	44	3,242	10,539,743 (smallest)
x^2+10x	144	150	165	36,075	1,301,766,376
x^2+11x	364	472	727	1,263,252	1,595,819,511,277
x^2+12x	140	172	212	52,928	2,802,008,321
x^2+13x	119	129	143	38,016	1,445,710,465
x^2+14x	66	96	135	25,395	645,261,556
x^2+15x	120	138	215	54,364	2,956,259,957
x^2+16x	75	99	141	45,177	2,041,684,162
x^2+17x	123	165	255	232,908	54,250,095,901
x^2+18x	70	74	88	12,968	168,402,449 (8th smallest)
x^2+19x	65	66	69	6,852	47,080,093 (4th smallest)
x^2+20x	84	96	115	24,261	589,081,342
x^2+21x	133	143	278	110,072	12,118,156,697
x^2+22x	165	177	317	165,423	27,368,408,236
x^2+23x	42	45	49	3,858	14,972,899 (3rd smallest)
x^2+24x	165	195	229	69,457	4,825,941,818
x^2+25x	17	20	42	9,300	86,722,501 (5th smallest)
x^2+26x	114	150	286	105,192	11,068,091,857
x^2+27x	105	111	132	29,178	852,143,491
x^2+28x	95	117	149	31,653	1,002,798,694
x^2+29x	91	111	145	36,792	1,354,718,233
x^2+30x	51	75	115	60,550	3,668,119,001
x^2+31x	39	60	96	14,922	223,128,667 (10th smallest)

Table for $x^2 + Bx$ where $1980 \leq B \leq 2000$.

g	x	y	z	w	$W(g(x); 4) \leq$
$x^2 + 1{,}980x$	1,683	2,145	2,915	25,524,829	651,567,434,640,662
$x^2 + 1{,}981x$	1,674	1,735	2,026	14,236,652	202,710,462,976,717
$x^2 + 1{,}982x$	1,248	1,495	1,731	6,882,723	47,385,517,451,716
$x^2 + 1{,}983x$	3,498	3,549	3,664	24,967,678	623,434,455,617,159
$x^2 + 1{,}984x$	860	975	2,585	12,424,497	154,392,775,905,058
$x^2 + 1{,}985x$	867	1,098	2,365	11,200,200	125,466,712,437,001
$x^2 + 1{,}986x$	1,900	2,432	2,908	19,712,552	388,623,855,480,977
$x^2 + 1{,}987x$	3,048	3,393	3,987	39,165,018	1,533,976,455,831,091
$x^2 + 1{,}988x$	508	738	1,194	6,489,996	42,132,950,192,065
$x^2 + 1{,}989x$	2,023	2,288	3,094	18,950,528	359,160,204,078,977
$x^2 + 1{,}990x$	1,364	1,610	2,100	13,163,856	173,313,300,862,177
$x^2 + 1{,}991x$	1,330	1,519	1,814	7,817,030	61,121,521,727,631
$x^2 + 1{,}992x$	975	1,065	1,871	10,120,498	102,444,639,800,021
$x^2 + 1{,}993x$	1,985	2,349	4,373	68,596,488	4,705,614,878,734,729 (9th largest)
$x^2 + 1{,}994x$	1,246	1,350	1,716	8,551,440	73,144,177,644,961
$x^2 + 1{,}995x$	891	1,185	1,464	10,543,450	111,185,372,085,251
$x^2 + 1{,}996x$	705	995	1,793	7,390,317	54,631,536,433,222
$x^2 + 1{,}997x$	1,081	1,136	1,391	8,040,026	64,658,074,012,599
$x^2 + 1{,}998x$	1,292	1,732	3,704	39,649,768	1,572,183,322,690,289
$x^2 + 1{,}999x$	1,235	1,757	2,789	14,633,322	214,163,364,766,363
$x^2 + 2{,}000x$	184	280	984	5,592,000	31,281,648,000,001

D Some Upper Bounds on $W(2x^2 + Bx; 4)$

The search for upper bounds on $W(2x^2 + Bx; 4)$ only worked for some values of B. We present all $0 \le B \le 2000$ for which B is odd and for which we found an upper bound on $W(2x^2 + Bx; 4)$.

g	x	y	z	w	$W(g(x); 4) \le$
$2x^2 + 57x$	3,969	4,035	4,295	38,199,155	2,918,353,062,779,886
$2x^2 + 95x$	707	758	1,008	14,365,638	412,744,475,029,699
$2x^2 + 171x$	11,907	12,105	12,885	343,792,395	236,386,480,508,171,596
$2x^2 + 285x$	2,121	2,274	3,024	129,290,742	33,432,228,781,682,599
$2x^2 + 399x$	27,783	28,245	30,065	1,871,758,595	7,006,961,222,744,427,456
$2x^2 + 455x$	3,320	3,663	4,170	39,229,128	3,077,866,816,534,009
$2x^2 + 511x$	2,772	3,367	6,282	131,899,720	34,795,139,672,913,721
$2x^2 + 627x$	43,659	44,385	47,245	4,622,097,755	5,834,090,064,188,269,204
$2x^2 + 805x$	1,210	1,303	2,920	87,446,025	15,293,684,970,651,376
$2x^2 + 855x$	5,548	7,087	13,262	530,042,423	561,890,393,545,693,524
$2x^2 + 1{,}011x$	5,164	6,568	9,889	318,517,859	202,907,575,025,443,212
$2x^2 + 1{,}153x$	12,705	12,726	12,970	352,488,525	248,496,726,932,620,576
$2x^2 + 1{,}199x$	8,245	8,710	9,748	221,108,291	97,778,017,806,722,272
$2x^2 + 1{,}295x$	14,030	14,355	22,244	1,162,712,925	2,703,804,197,637,349,126
$2x^2 + 1{,}301x$	25,622	26,105	28,172	1,638,880,116	5,371,858,201,423,377,829
$2x^2 + 1{,}365x$	9,960	10,989	12,510	353,062,152	249,306,248,279,579,689
$2x^2 + 1{,}459x$	954	1,174	1,379	58,465,486	6,836,511,407,576,467
$2x^2 + 1{,}545x$	11,298	11,815	12,860	425,440,418	361,999,755,841,475,259
$2x^2 + 1{,}685x$	10,695	10,968	11,570	289,144,125	167,209,137,251,881,876
$2x^2 + 1{,}753x$	3,586	5,236	8,232	181,967,394	66,224,583,947,144,155
$2x^2 + 1{,}851x$	50,031	51,441	55,164	6,379,649,159	7,612,882,297,751,201,408
$2x^2 + 1{,}913x$	2,261	3,366	5,324	81,424,299	13,259,988,699,966,790

E Some Upper Bounds on $W(3x^2 + Bx; 4)$

The search for upper bounds on $W(3x^2 + Bx; 4)$ only worked for some values of B. We present all $0 \leq B \leq 2000$ for which B is not divisible by 3 and for which we found an upper bound on $W(3x^2 + Bx; 4)$.

g	x	y	z	w	$W(g(x); 4) \leq$
$3x^2 + x$	42,273	42,660	43,375	5,738,872,934	6,570,267,294,984,419,923
$3x^2 + 143x$	13,244	13,332	13,442	554,651,696	922,915,590,942,221,777
$3x^2 + 172x$	4,452	4,712	5,189	88,862,311	23,689,546,233,099,656
$3x^2 + 200x$	1,896	2,204	5,004	115,177,723	39,797,746,661,938,788
$3x^2 + 235x$	11,155	11,270	11,610	583,594,418	1,021,747,471,306,964,403
$3x^2 + 274x$	9,322	11,610	16,903	1,125,018,929	3,797,003,080,080,107,670
$3x^2 + 344x$	8,904	9,424	10,378	355,449,244	379,032,617,455,054,545
$3x^2 + 361x$	3,540	4,658	7,703	397,333,094	473,620,906,200,085,443
$3x^2 + 400x$	3,792	4,408	10,008	460,710,892	636,763,762,306,663,793
$3x^2 + 407x$	2,806	3,401	6,131	122,898,626	45,312,266,837,804,411
$3x^2 + 412x$	2,077	2,829	5,839	392,773,686	462,813,667,064,838,421
$3x^2 + 520x$	7,616	9,244	12,716	515,261,395	796,483,183,467,963,476
$3x^2 + 556x$	9,400	9,408	9,451	273,674,799	224,693,838,986,259,448
$3x^2 + 592x$	15,744	16,472	17,944	994,061,387	2,964,474,711,857,432,412
$3x^2 + 643x$	50,932	51,357	52,351	8,273,167,696	2,421,731,687,255,606,001
$3x^2 + 688x$	17,808	18,848	20,756	1,421,796,976	6,064,520,901,084,553,217
$3x^2 + 725x$	3,172	3,185	3,278	34,869,750	3,647,723,675,756,251
$3x^2 + 728x$	16,744	17,360	18,928	1,174,742,491	4,140,060,615,695,188,692
$3x^2 + 797x$	2,847	3,082	3,524	148,907,272	66,520,245,642,541,737
$3x^2 + 814x$	5,612	6,802	12,262	491,594,504	724,995,869,246,944,305
$3x^2 + 932x$	1,820	2,229	2,799	37,745,311	4,274,160,686,090,016
$3x^2 + 1{,}085x$	1,190	1,344	1,540	10,401,450	324,581,771,880,751
$3x^2 + 1{,}087x$	9,800	9,909	11,434	604,108,526	1,094,841,990,223,645,791
$3x^2 + 1{,}112x$	18,800	18,816	18,902	1,094,699,196	3,595,100,206,474,645,201

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