Distinct Rado's Theorem

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In the examples of Rado, and the proof of it, it could well be that some of the x_i's are the same. Can we always avoid this?

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with all of the x_i's distinct.

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VOTE T or F or UNKNOWN TO SCIENCE.

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We need a condition on $(b_1, \ldots, b_n)$.

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Correct Distinct Rado Theorem

Thm Let $b_1, \ldots, b_k \in \mathbb{Z}$. TFAE

- Some subset of the b_i's sums to 0 AND there exists distinct $\lambda_1, \ldots, \lambda_k$ such that $\sum_{i=1}^{k} \lambda_i b_i = 0$.
- For all c, for all $\chi\colon \mathbb{N} \to [c]$ there exists mono solution to

$$b_1 x_1 + \cdots + b_k x_k = 0$$

  with all of the x_i's distinct.

Motivating Example

$$x_1 - x_2 + x_3 + 3x_4$$

Note that $1 \times 1 - 1 \times 12 + 1 \times 2 + 3 \times 3 = 0$.
So $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (1, 12, 2, 3)$.

Motivating Example

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Note that $1 \times 1 - 1 \times 12 + 1 \times 2 + 3 \times 3 = 0$.
So $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (1, 12, 2, 3)$.
By EVDW, for any col of $\mathbb{N}$ there exists a, d such that

$$a, a+d, a+2d, a+3d, a+4d, a+5d, d \text{ are same color}$$

Let
$x_1 = a + d$
$x_2 = a + 5d$
$x_3 = d$
$x_4 = d$
This is a solution, though notice that $x_3 = x_4$.

$x_1 - x_2 + x_3 + 3x_4 \ (\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (1, 12, 2, 3)$

Mono Solution is:

$x_1 = a + d$

$x_2 = a + 5d$

$x_3 = d$

$x_4 = d$

What if offsets of x_1, x_2, x_3, x_4 by mults of D were same color?

$x_1' = x_1 + M_1 D$

$x_2' = x_2 + M_2 D$

$x_3' = x_3 + M_3 D$

$x_4' = x_4 + M_4 D$

Need (M_1, M_2, M_3, M_4) such that (x_1', x_2', x_3', x_4') is a sol. Use λ_i.

$$(x_1 + \lambda_1 D) - (x_2 + \lambda_2 D) + (x_3 + \lambda_3 D) + 3(x_4 + \lambda_4 D) =$$

$$(x_1 - x_2 + x_3 + 3x_4) + (\lambda_1 - \lambda_2 + \lambda_3 + 3\lambda_4)D = 0 + 0 = 0$$

Key Theorem We Need

$(\forall b_1, \ldots, b_n, c, M)(\exists L)$ $\forall$ c-coloring $\chi \to [L] \to [c]$ $\exists$ $x_1, \ldots, x_n, D \in [L]$ such that

1. $b_1 x_1 + \cdots + b_n x_n = 0$
2. The following are all the same color.

$$\begin{array}{ccccccc}
x_1 - MD, & \ldots, & x_1 - D, & x_1, & x_1 + D, & \ldots, & x_1 + MD \\
x_2 - MD, & \ldots, & x_2 - D, & x_2, & x_2 + D, & \ldots, & x_2 + MD \\
\vdots & & \vdots & \vdots & \vdots & & \vdots \\
x_n - MD, & \ldots, & x_n - D, & x_n, & x_n + D, & \ldots, & x_n + MD.
\end{array}$$

We first prove a Lemma.

Lemma One (Nothing to do w/equations)

$(\forall c, R, X)(\exists L)$ $\forall$ c-coloring $\chi \to [L] \to [c]$ $\exists$ a, D such that

$$\begin{array}{lllllll}
\chi(a - XD) = & \chi(a-(X-1)D) & \doteq & \chi(a) & \doteq & \chi(a+XD) \\
\chi(2(a - XD)) = & \chi(2(a-(X-1)D)) & \doteq & \chi(2a) & \doteq & \chi(2(a+XD)) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\chi(R(a - XD)) = & \chi(R(a-(X-1)D)) & \doteq & \chi(Ra) & \doteq & \chi(R(a+XD))
\end{array}$$

(Note- the rows may be diff colors.)

Pf We color $\mathbb{N}$. Details of finding bounds is an exercise.
Given $\chi \colon \mathbb{N} \to [c]$ we devise another coloring
$\chi* \colon \mathbb{N} \to [c]^R$ via $\chi^*(n) = (\chi(n), \ldots, \chi(Rn))$.
Apply VDW and see next slide.

Lemma One Cont

Pf We color $\mathbb{N}$. Details of finding bounds is an exercise.
Given $\chi\colon \mathbb{N}\to[c]$ we devise another coloring
$\chi^*\colon \mathbb{N}\to[c]^R$ via $\chi^*(n) = (\chi(n), \ldots, \chi(Rn))$.
Apply VDW to get a, D such that

$$\chi^*(a-XD) = \chi^*(a-(X-1)D) = \cdots = \chi^*(a) = \cdots = \chi^*(a+XD).$$

Lemma One Cont

$$\chi^*(a - XD) = \chi^*(a - (X-1)D) = \cdots = \chi^*(a) = \cdots = \chi^*(a + XD).$$

1st coord yields $\chi(a - XD) = \chi(a - (X-1)D) = \cdots = \chi(a + XD)$.
2nd coord yields
$\chi(2(a - XD)) = \chi(2(a - (X-1)D)) = \cdots = \chi(2(a + XD))$.
Etc. So we get:

$$\begin{array}{llllll}
\chi(a - XD) = & \chi(a - (X-1)D) & \doteq & \chi(a) & \doteq & \chi(a + XD) \\
\chi(2(a - XD)) = & \chi(2(a - (X-1)D)) & \doteq & \chi(2a) & \doteq & \chi(2(a + XD)) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\chi(R(a - XD)) = & \chi(R(a - (X-1)D)) & \doteq & \chi(Ra) & \doteq & \chi(R(a + XD))
\end{array}$$

QED

Key Theorem We Need, And its Proof!

$(\forall b_1, \ldots, b_n, c, M)(\exists L)$ $\forall$ c-coloring $\chi \rightarrow [L] \rightarrow [c]$ $\exists$ $x_1, \ldots, x_n, d \in [L]$ such that

1. $b_1x_1 + \cdots + b_nx_n = 0$
2. The following are all the same color.

$$\begin{array}{ccccccc} x_1 - MD, & \ldots, & x_1 - D, & x_1, & x_1 + D, & \ldots, & x_1 + MD \\ x_2 - MD, & \ldots, & x_2 - D, & x_2, & x_2 + D, & \ldots, & x_2 + MD \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ x_n - MD, & \ldots, & x_n - D, & x_n, & x_n + D, & \ldots, & x_n + MD. \end{array}$$

Pf Let $\chi \colon \mathbb{N} \rightarrow [c]$.

Proof of Key Theorem

We apply Lemma with $R = R(b_1, \ldots, b_n; c)$ to get:

$$
\begin{array}{llllll}
\chi(a - XD) = & \chi(a - (X-1)D) & \doteq & \chi(a) & \doteq & \chi(a + XD) \\
\chi(2(a - XD)) = & \chi(2(a - (X-1)D)) & \doteq & \chi(2a) & \doteq & \chi(2(a + XD)) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\chi(R(a - XD)) = & \chi(R(a - (X-1)D)) & \doteq & \chi(Ra) & \doteq & \chi(R(a + XD)
\end{array}
$$

(Note- the rows may be diff colors.)
Let $\chi^* \colon [R] \rightarrow [c]$ be $\chi^*(x) = \chi(xa)$, so the col of the row.
By def of R, exists $f_1, \ldots, f_n$ such that

1. $\sum_{i=1}^{n} b_i f_i = 0$. Hence $\sum_{i=1}^{n} b_i(af_i) = a\sum_{i=1}^{n} b_i f_i = 0$.
2. $\chi^*(f_1) = \chi^*(f_2) = \cdots = \chi^*(f_n)$.
   By the def of χ^*, $\chi(af_1) = \cdots = \chi(af_n)$.

All of These Rows The Same Color!

We have that the following are *all* the same color:

$$\begin{array}{cccccc}
(a-XD)f_1, & (a-(X-1)D)f_1, & \cdots, & af_1, & \cdots, & (a+XD)f_1 \\
(a-XD)f_2, & (a-(X-1)D)f_2, & \cdots, & af_2, & \cdots, & (a+XD)f_2 \\
(a-XD)f_3, & (a-(X-1)D)f_3, & \cdots, & af_3, & \cdots, & (a+XD)f_3 \\
\vdots & \vdots & & \vdots & & \vdots \\
(a-XD)f_n, & (a-(X-1)D)f_n, & \cdots, & af_n, & \cdots, & (a+XD)f_n.
\end{array}$$

For all i, $1 \le i \le n$ let $x_i = af_i$. We rewrite the above:

$$\begin{array}{cccccc}
x_1 - f_1XD, & x_1 - f_1(X-1)D, & \cdots, & x_1, & \cdots, & x_1 + f_1XD \\
x_2 - f_2XD, & x_2 - f_2(X-1)D, & \cdots, & x_2, & \cdots, & x_2 + f_2XD \\
x_3 - f_3XD, & x_3 - f_3(X-1)D, & \cdots, & x_3, & \cdots, & x_3 + f_3XD \\
\vdots & \vdots & & \vdots & & \vdots \\
x_n - f_nXD, & x_n - f_n(X-1)D, & \cdots, & x_n, & \cdots, & x_n + f_nXD.
\end{array}$$

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x_2 - f_2 XD, & x_2 - f_2(X-1)D, & \cdots, & x_2, & \cdots, & x_2 + f_2 XD \\
x_3 - f_3 XD, & x_3 - f_3(X-1)D, & \cdots, & x_3, & \cdots, & x_3 + f_3 XD \\
\vdots & \vdots & & \vdots & & \vdots \\
x_n - f_n XD, & x_n - f_n(X-1)D, & \cdots, & x_n, & \cdots, & x_n + f_n XD.
\end{array}$$

Want same offset, not $f_1 D$, $f_2 D$, etc.

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x_3 - f_3 XD, & x_3 - f_3(X-1)D, & \cdots, & x_3, & \cdots, & x_3 + f_3 XD \\
\vdots & \vdots & & \vdots & & \vdots \\
x_n - f_n XD, & x_n - f_n(X-1)D, & \cdots, & x_n, & \cdots, & x_n + f_n XD.
\end{array}$$

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Just take X large enough and thin out each row. Details omitted.