# Folkman's Theorem

**Exposition by William Gasarch**

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More generally, we want **all** non-empty sums are the same color.

# $b_1 < b_2$ Theorem

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Let $T = 3c$ (this is prob not optimal).

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**Note** $b_1 < b_2$ thm follows from Schur, but we wanted elt proof.

# $b_1 < b_2$ Theorem With Divisibility

**Thm** $(\forall c, d)(\exists T = T(c)d)$ st $\forall$ $\mathrm{COL} : [T] \rightarrow [c]$ $\exists b_1 < b_2$ st $\mathrm{COL}(b_2) = \mathrm{COL}(b_1 + b_2)$ AND $b_1 < b_2$ AND $b_1, b_2 \equiv 0 \pmod{d}$.

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WRITE DOWN WHAT $T(c, d)$ MEANS FOR LATER USE.

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We later show general case of $b_1 < \cdots < b_n$.

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Don't know $d$.

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Don't know $d$. **Boo!**

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Don't know $d$. **Boo!** But know $d \le \frac{W(k,c)}{k} \le W(k,c)$.

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**Upshot** Need $b_2' + b_1' \le k$. Take $k = 2T(c) = 6c$.

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$$U(c) = 3cW(6c, c) + W(6c, c) = (3c+1)W(6c, c).$$

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**Thm** $(\forall n, c)(\exists U = U(n,c))$ st $\forall$ COL : $[U] \rightarrow [c]$ $\exists b_1 < \cdots < b_n$ st

2) $(\forall I \subseteq \{1\})[\mathrm{COL}(b_2 + \sum_{i \in I} b_i)$ same color$]$.

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Will prove on next slides.

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Fix $c$. $U$ is TBD. Assume there $\mathrm{COL}\colon [U] \rightarrow [c]$.

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Done!

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Proof on Next Slide.

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Look at $c_1, \ldots, c_{2n-1}$. $n$ of them are the same color,say $R$.
Call those $n$ $x_1, \ldots, x_n$.
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