DTIME, P, EXP, and of Course NP

Exposition by William Gasarch—U of MD

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We first look at some problems of interest.

Problems of Interest

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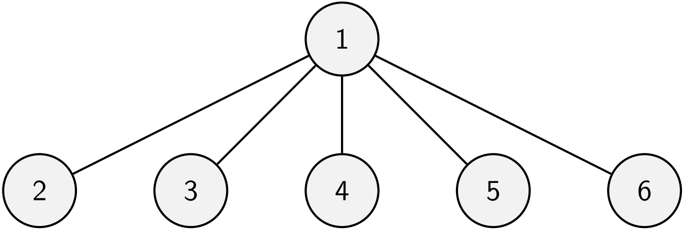

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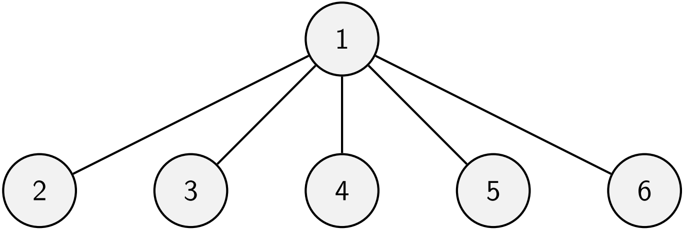

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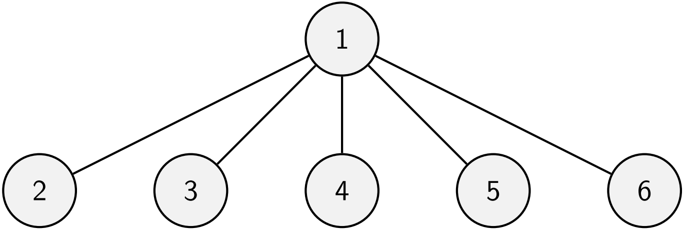

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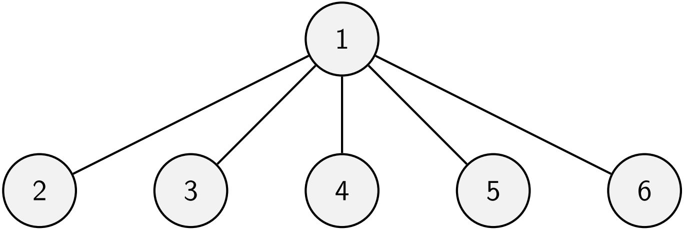

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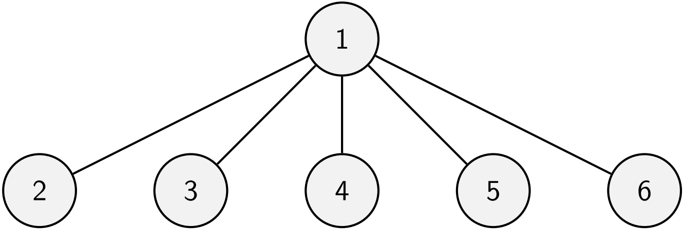

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In the above graph $\mathrm{degh}(1) = 5$ and
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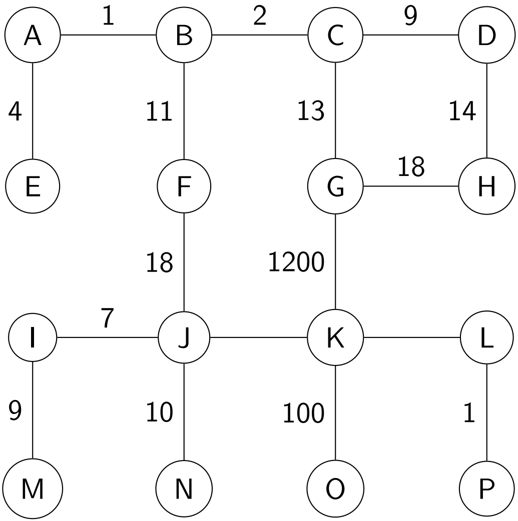
Weight of (x, y) is price of the flight. (Cost is symmetric.)

Another Example of a Weighted Graph



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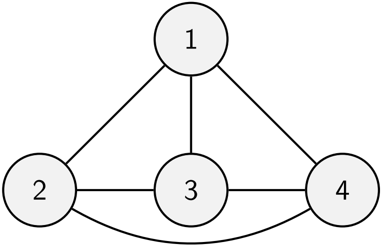

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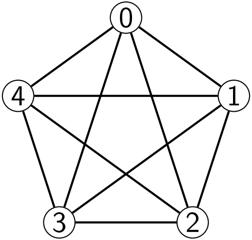

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2. A model of Computation. A statement like **EUL can be solved in time $O(n)$** needs to say what device we are computing on.

Formalizing Representation and Computation

Exposition by William Gasarch—U of MD

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Speed for Engineers

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2. A graph is represented by an adjacency matrix. An n-node graph is an n^2-long string.
3. A **set of graphs** (like HAMC) is a set of strings, all of square length, all interpreted as an adjacency matrix for a graph.

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We Sometimes Cheat We may take the length of a graph to be the number of vertices. These notions of length are poly-related to the actual length and hence is fine for our purposes.

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2. Turing machines compute with discrete steps so one can talk about how many steps a computation takes.

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   Not practical. But that cleverness can probably be exploited to get a practical algorithm.

Back to our Problems

Exposition by William Gasarch—U of MD

HAM, EUL, CLIQ All Walk into a Bar

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Why is this interesting?

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HAM, EUL, CLIQ are similar.

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- If $x \in A$ then there is a SHORT (poly in $|x|$) proof of this fact, namely y, such that x can be VERIFIED in poly time. So if I wanted to convince you that $x \in A$, I could give you y. You can verify $(x, y) \in B$ easily and be convinced.
- If $x \notin A$ then there is NO proof that $x \in A$.

Note HAM, EUL, CLIQ are all in NP.

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So Why Is This Important

Def of NP-Complete

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Cook and Levin in the early 1970's showed that SAT, a problem in logic, was NPC. They coded TM's into formulas. We won't do that here.

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The good money says that None are in Poly Time.

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7. Games and Puzzles
8. Others

History: HAM and EUL

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The theory of NP-completeness enabled mathematicians to **state** what they wanted rigorously ($\mathrm{HAM} \in \mathrm{P}$) and also gave the basis for proving likely it **cannot** be done (since HAM is NP-Complete).

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3. $\mathrm{P} \neq \mathrm{NP}$ has great explanatory power. See next slide.

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Set Cover Given n and $S_1, \ldots, S_m \subseteq \{1, \ldots, n\}$ find the least number of sets S_i's that **cover** $\{1, \ldots, n\}$.

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3. These two proofs have nothing to do with each other yet give matching upper and lower bounds.
4. There are many other approx problems where $\mathrm{P} \neq \mathrm{NP}$ explains why they cannot be improved.

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    1.2 IF $\mathrm{P} \neq \mathrm{NP}$ this will not be proven until the year 2525.
2. $\mathrm{P} \neq \mathrm{NP}$. In fact, SAT requires $2^{\Omega(n)}$ time.

What Do Theorists Think of P vs NP?

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	$P \neq NP$	$P = NP$	Ind	DK	other
2002	61 (61%)	9 (9%)	4 (4%)	22 (22%)	7 (7%))
2012	126 (83%)	12 (9%)	5 (3%)	1 (0.66%)	8 (5.1%)
2019	109 (88%)	15 (12%)	0	0	0