**Closure of Regular Langs Under Union, Intersection, Complementation, and Projection**
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# 1 Introduction

We give the constructions that show sketch the proof that all if $L_1$ and $L_2$ are regular and $L_1 \cap L_2$, $L_1 \cup L_2$, $\overline{L}$, and $proj(L)$ (which we will define) are regular.

**Def 1.1** A DFA is a tuple $(Q, \Sigma, \delta, s, F)$ where $\delta : Q \times \Sigma \rightarrow Q$.

We define running a DFA $M$ on a string $x$ in the obvious way. If the DFA ends in a state in $F$ then $x$ is accepted. Otherwise its rejected.

# 2 Closure Under Intersection

**Theorem 2.1** *If $L_1$ and $L_2$ are regular then $L_1 \cap L_2$ is regular.*

**Proof:**
Let $M_1 = (Q_1, \Sigma, \delta_1, s_1, F_1)$ be the DFA for $L_1$. Let $M_2 = (Q_2, \Sigma, \delta_2, s_2, F_2)$ be the DFA for $L_2$.

We define a DFA for $L_1 \cap L_2$. Let $M = (Q_1 \times Q_2, \Sigma, \delta, (s_1, s_2), F_1 \times F_2)$ where $\delta$ is defines by, for $(q_1, q_2) \in Q_1 \times Q_2$ and $\sigma \in \Sigma$,

$$\delta((q_1, q_2), \sigma) = (\delta_1(q_1, \sigma), \delta_2(q_2, \sigma)).$$

The intuition is that the DFA $M$ runs $M_1$ and $M_2$ at the same time. If both end up in $F_1 \times F_2$ then both $M_1$ and $M_2$ accepted. ∎

# 3 Closure Under Union

**Theorem 3.1** *If $L_1$ and $L_2$ are regular then $L_1 \cup L_2$ is regular.*

**Proof:**
Let $M_1 = (Q_1, \Sigma, \delta_1, s_1, F_1)$ be the DFA for $L_1$. Let $M_2 = (Q_2, \Sigma, \delta_2, s_2, F_2)$ be the DFA for $L_2$.

We define a DFA for $L_1 \cup L_2$. Let $M = (Q_1 \times Q_2, \Sigma, \delta, (s_1, s_2), F_1 \times Q_2 \cup Q_1 \times F_2)$ where $\delta$ is defines by, for $(q_1, q_2) \in Q_1 \times Q_2$ and $\sigma \in \Sigma$,

$$\delta((q_1, q_2), \sigma) = (\delta_1(q_1, \sigma), \delta_2(q_2, \sigma)).$$

The intuition is that the DFA $M$ runs $M_1$ and $M_2$ at the same time. If $M_1$ ends up in $F_1$ then we accept (independent of what $M_2$ does), and if $M_2$ ends up in $F_2$ then we accept (independent of what $M_1$ does). ∎

# 4 Closure Under Complementation

**Theorem 4.1** *If $L$ is regular then $\overline{L}$ is regular.*

**Proof:**
Let $M = (Q, \Sigma, \delta, s, F)$ be the DFA for $L$.
We define a DFA for $\overline{L}$. Let $M' = (Q, \Sigma, \delta, s, Q - F)$ (recall that $Q - F = \{q \mid q \in Q \wedge q \notin F\}$.
The intuition is that the DFA $M'$ runs $M$ but does the opposite when it comes to accepting. ∎

# 5 Closure Under Complimentation

To Compliment a DFA you say
*DFA, I admire your states!*

# 6 NDFA's and DFA's

Recall the definition of an NDFA:

**Def 6.1** An NDFA is a tuple $(Q, \Sigma, \Delta, s, F)$ where $\Delta : Q \times (\Sigma \cup e) \rightarrow 2^Q$. (Recall that $2^Q$ is the powerset of $Q$.

We DO NOT define running an NDFA $M$ on a string $x$. Instead we say that an NDFA accepts $x$ if SOME way of running the machine ends up in a state in $F$.

**Theorem 6.2** *If $L$ is accepted by an NDFA then there exists a DFA such that accepts $L$.*

**Proof:** Let $M = (Q, \Sigma, \Delta, s, F)$ be the NDFA for $L$.
We define a DFA for $L$. Let $M' = (2^Q, \Sigma, \delta, s, \mathcal{F})$ where for $A \in 2^Q$ and $\sigma \in \Sigma$,

$$\delta(A, \sigma) = \bigcup_{q \in A} \Delta(e^a q e^b, \sigma)$$

(The $e^a$ and $e^b$ are strings of the empty string.)

$$\mathcal{F} = \{A \mid A \cap F \neq \emptyset\}$$

The intuition is that the DFA $M'$ runs ALL possibilities for $M$. If SOME possibility ends up accepting, then accept. ∎

## 7 Closure under Projection

**Notation 7.1** Let$\Sigma = \{0,1\}^n$. Note that each element of $\Sigma$ is itself a string of $n$ bits. If $x \in \Sigma^*$ then $proj(x)$ is what you get by taking each symbol in $x$ and chopping off the last bit. So if $x \in (\{0,1\}^n)^*$ then $proj(x) \in (\{0,1\}^{n-1})^*$. If $L \subseteq (\{0,1\}^n)^*$ then

$$proj(L) = \{proj(x) \mid x \in L\}.$$

**Theorem 7.2** *If $L$ is regular than $proj(L)$ is regular.*

**Proof:** Let $M = (Q, (\{0,1\}^n), \delta, s, F)$ be the DFA for $L$.

We define an NDFA for $L$. Let $M' = (Q, \{0,1\}^{n-1}, \Delta, s, F)$ . For $q \in Q$ and $\sigma \in \{0,1\}^{n-1}$

$$\Delta(q,\sigma) = \{\delta(q,\sigma 0), \delta(q,\sigma 1)\}.$$

∎

## 8 Closure under Concatenation

**Theorem 8.1** *If $L_1$ and $L_2$ are regular then $L_1L_2$ is regular.*

**Proof:**

Let $M_1 = (Q_1, \Sigma, \delta_1, s_1, F_1)$ be the DFA for $L_1$. Let $M_2 = (Q_2, \Sigma, \delta_2, s_2, F_2)$ be the DFA for $L_2$. By relabelling we can assume $Q_1 \cap Q_2 = \emptyset$.

We define an NDFA for $L_1L_2$. By Theorem **??** we could then obtain a DFA for $L_1L_2$.

Let $M = (Q_1 \cup Q_2, \Sigma, \delta, s_1, F_2)$ where $\delta$ is defines as follows:

- If $q_1 \in Q_1$ and $\sigma \in \Sigma$ then $\delta(q_1, \sigma) = \delta_1(q_1, \sigma)$.
- If $q_2 \in Q_2$ and $\sigma \in \Sigma$ then $\delta(q_2, \sigma) = \delta_2(q_2, \sigma)$.
- If $f_1 \in F_1$ then $\delta(f_1, e) = s_2$

The intuition is that the NDFA $M$ runs $M_1$ and then nondeterministically hops to $M_2$. But the hop must be from $f_1 \in F_1$ to $s_2 \in Q_2$ and then the $M_2$ must accept, so if $w$ is accepted there must be SOME WAY to divide it $w = xy$ where $x \in L_1$ and $y \in L_2$. ∎

## 9 Closure Under ∗

**Theorem 9.1** *If $L$ is regular then $L^*$ is regular.*

**Proof:**

Let $M = (Q, \Sigma, \delta, s, F)$ be the DFA for $L$.

We define an NDFA for $L^*$. LEAVE AS AN EXERCISE.

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