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The Probabilistic Method: Sum-Free Sets

Exposition by William Gasarch

Sum Free Set Problem

Definition: A set of numbers A is **sum-free** if there is NO $x, y, z \in A$ such that $x + y = z$.

Example: Let $y_1, \ldots, y_m \in (1/3, 2/3)$ (so they are all between $1/3$ and $2/3$). Note that $y_i + y_j > 2/3$, hence $y_i + y_j \notin \{y_1, \ldots, y_m\}$.

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Proof left to the reader.

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