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On a Bound of Graham and Spencer for a Graph-Coloring Constant*

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$\alpha(k, l; r)$ denotes the smallest number of vertices in any graph G that has the properties;

(1) G contains no complete subgraph on l vertices,

(2) in any r-coloring of the edges of G, some complete subgraph on k vertices is monochromatic. We show $\alpha(3, 5; 2) \leqslant 18$, improving a bound due to Graham and Spencer [4].

1. INTRODUCTION

Denote by $S(k, l; r)$ the following statement: There exists a graph G having the properties

(1) G contains no complete subgraph on l vertices,

(2) if the edges of G are colored anyhow using r colors, then some complete subgraph on k vertices has all of its edges the same color, i.e., is monochromatic.

Denote by $R(k_1, k_2, ..., k_r; 2)$ that Ramsey number that is the smallest integer n such that, in any r-coloring of the edges of K_n, the complete graph on n vertices, some K_{k_i} has all of its edges the i-th color, for some $i (1 \leqslant i \leqslant r)$. In particular, when $k_1 = k_2 = \cdots = k_r = k$, we denote the corresponding Ramsey number by $R_r(k; 2)$.

It follows at once from Ramsey's theorem that, for fixed k and r, $S(k, l; r)$ is true for $l \geqslant R_r(k; 2) + 1$. It is well known that $R_2(3; 2) = 6$, so that $S(3, 7; 2)$ holds. Erdös and Hajnal [1] asked whether $S(3, 6; 2)$ holds, and van Lint (unpublished) gave an affirmative answer. Posa

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(unpublished) first showed $S(3. 5; 2)$ true, and Folkman [2] showed $S(k, k+1; 2)$ true for all $k \geqslant 3$, and so $S(3, 4; 2)$ true as a special case. It was conjectured by Folkman, and also by Erdös and Hajnal, that $S(k, k+1; r)$ is true for all k and r.

Let us denote by $\mathscr{G}(k, l; r)$ the class of all graphs G (if any) that possess properties (1) and (2) above. A further problem is that of determining the smallest number $\alpha(k, l; r)$ of vertices of any graph in $\mathscr{G}(k, l; r)$. The only results known are:

$\alpha(3, 6; 2) = 8$ (Graham [3]),

$10 \leqslant \alpha(3, 5; 2) \leqslant 23$ (Shen Lin [8], Graham and Spencer [4]).

In [4], Graham and Spencer conjecture that $\alpha(3, 5; 2) = 23$, though, as they admit, this was without much evidence. Our main objective here is to show $\alpha(3, 5; 2) \leqslant 18$.

2. n-Chromatic Number and a Theorem of Sachs

For $n \geqslant 2$, the n-chromatic number $\chi_n(G)$ of a graph G is the smallest number of classes among which the vertices of G may be distributed in such a way that no n mutually adjacent vertices lie in the same class.

The following is a theorem of Sachs [7]:

Theorem (Sachs). *Given positive integers n, s ($n \geqslant 2$) there exists a graph H with the properties*:

(1) *H does not contain a complete subgraph on $n + 1$ vertices,*

(2) $\chi_n(H) = s$.

In fact, this is a special case of Folkman's theorem 2 [2], but it will be sufficient for our needs.

Denote by $\mathscr{H}(n, s)$ the class of graphs possessing properties (1) and (2) of the above theorem, and by $h(n, s)$ the smallest number of vertices in any member of $\mathscr{H}(n, s)$.

Lemma 1. $h(3, 3) \leqslant 17$.

Proof. We construct a graph H on 17 vertices as follows: label the vertices $V_1, V_2, \ldots, V_{17}$, and join vertices V_i, V_j ($1 \leqslant i < j \leqslant 17$) by an edge if and only if $j - i$ is a quadratic residue (mod 17), i.e., one of 1, 2, 4, 8, 9, 13, 15, 16. It is well known (see, e.g., [5]) that H contains no complete subgraph on 4 vertices. We claim that $\chi_3(H) = 3$.

Suppose that there exists a 2-coloring (in red and blue, say) of the vertices of H so that no three adjacent vertices have the same color. Since 17 is odd, there are two similarly colored vertices V_i, V_j with $j - i \equiv 1 \pmod{17}$. Now, it is clear that H is a point-symmetric graph (see [6]) so that, without loss of generality, we can assume V_1 and V_2 are red, say. Then V_3, V_{17} and V_{10} are blue. Now, at least one of V_9, V_{11} is red, and, without loss of generality, we can assume V_9 red. Then V_5 is blue, V_4 red, V_6 blue, and V_7 red. But now V_8 cannot be blue, otherwise V_6, V_8, V_{10} are three mutually adjacent blue vertices. Nor can V_8 be red, otherwise V_7, V_8, V_9 are three mutually adjacent red vertices. Hence we have a contradiction. Further, the partition $\{V_1, V_4, V_7, V_{10}, V_{13}, V_{16}\}$, $\{V_2, V_5, V_8, V_{11}, V_{14}, V_{17}\}$, $\{V_3, V_6, V_9, V_{12}, V_{15}\}$ shows that

$$H \in \mathscr{H}(3, 3).$$

3. The Main Result

Theorem. $\alpha(3, 5; 2) \leqslant 18$.

In order to prove the theorem we shall need a further definition and a lemma.

The *join* $G_1 + G_2$ of two graphs G_1 and G_2 is the graph whose vertex set is the union of the vertex sets of G_1, G_2, and whose edge set is the union of the edge sets of G_1, G_2, together with the set of all possible edges joining a vertex of G_1 to a vertex of G_2.

Lemma 2. *Let $l = R(k, k, \ldots, k, k - 1; 2) + 2$, where there are exactly $r - 1$ k's in the parameter list of the Ramsey number, and let*

$$H \in \mathscr{H}(l - 2, r + 1).$$

Then, if G is the join of H and a single vertex V, $G \in \mathscr{G}(k, l; r)$.

Proof. First, $H \in \mathscr{H}(l - 2, r + 1)$ implies that H does not contain K_{l-1} as a subgraph, which in turn implies that G does not contain K_l as a subgraph.

Suppose that we have an r-coloring of the edges of G which contains no monochromatic K_k. Attach to each vertex of H the color of the edge joining that vertex to the vertex V. Since $\chi_{l-2}(H) = r + 1$, H must contain a set S of $l - 2 = R(k, k, \ldots, k, k - 1; 2)$ mutually adjacent vertices all of the same color, C_1 say, But in order that G should contain no monochromatic K_k, the subgraph induced by S cannot contain

(1) $k - 1$ vertices all joined by edges of color C_1,

(2) k vertices all joined by edges of any one other color.

This contradicts the definition of $R(k, k, \dots, k, k - 1; 2)$, and the lemma is proved.

Proof of Theorem. Let H be the graph of Lemma 1, V a single vertex, and $G = H + V$. Then $H \in \mathscr{H}(3, 3)$, and, taking $l = 5$, $r = 2$, $k = 3$, we see that all the conditions of Lemma 2 are satisfied. Hence $G \in \mathscr{G}(3, 5; 2)$, and since G has 18 vertices, the theorem is proved.

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(i) The theorem of section 2 attributed to Sachs was first proved by Erdös and Rogers [9].

(ii) Posa (unpublished) was first to establish the truth of $S(3, 6; 2)$ and $S(3, 5; 2)$. The method used by Posa to establish $S(3, 5; 2)$ was essentially that of the present paper, but Posa used only the fact that the class of graphs $\mathscr{H}(3, 3)$ is non-empty. He gave no upper bound for $h(3, 3)$.

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