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On Ramsey Numbers and $\mathbf{K}_r$-Coloring of Graphs

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Let $\mathbf{K}_r = (a_1, a_2, ..., a_r)$, a_i's integers $\geqslant 2$, $r \geqslant 1$. By a $\mathbf{K}_r$-coloring of a graph G we mean a coloring of the edges of G by distinct colors $c_1, c_2, ..., c_r$ such that there are no complete subgraphs on a_i vertices whose edges are all colored in color c_i ($i = 1, 2, ..., r$). In this paper, we consider $\mathbf{K}_r$-colorings of the set H_l^v of all graphs on v vertices which do not contain a complete subgraph on l vertices. The interesting cases are those with $v \geqslant R(\mathbf{K}_r) \geqslant l$, where $R(\mathbf{K}_r)$ is the Ramsey number associated with $\mathbf{K}_r$. Furthermore, we also construct a family of graphs $\in H_l^v$ with minimum v which cannot be $\mathbf{K}_r$-colored.

I. INTRODUCTION: SOME RESULTS ON RAMSEY NUMBERS

Let $\mathbf{K}_r = (a_1, a_2, ..., a_r)$, a_i's integers $\geqslant 2$, $r \geqslant 1$. By a $\mathbf{K}_r$-coloring of a graph G we mean a coloring of the edges of G by distinct colors $c_1, c_2, ..., c_r$ such that there are no complete subgraphs on a_i vertices whose edges are all colored in color c_i ($i = 1, 2, ..., r$). From Ramsey's theorem [1], there exist integers $R(\mathbf{K}_r)$ such that, if G is any complete graph on $R(\mathbf{K}_r)$ or more vertices, then G cannot be $\mathbf{K}_r$-colored while all complete graphs on $R(\mathbf{K}_r) - 1$ or fewer vertices can be $\mathbf{K}_r$-colored. The numbers $R(\mathbf{K}_r)$ are known as Ramsey numbers with parameters $\mathbf{K}_r$.

Trivially, we have

(1) $R(a_1) = a_1$.

(2) $R(a_1, a_2, ..., a_r, 2) = R(a_1, a_2, .., a_r)$.

(3) *If* $\mathbf{K}_r' = (b_1, b_2, ..., b_r)$ where $(b_1, b_2, ..., b_r)$ is any rearrangement of $(a_1, a_2, ..., a_r)$, then $R(\mathbf{K}_r') = R(\mathbf{K}_r)$.

Except for those cases, very few other Ramsey numbers are known. We list them in Table I [2, 3, 6].

In view of relations (1) and (2), we shall assume the a_i's $\geqslant 3$ and $r \geqslant 2$ in $\mathbf{K}_r$.

Let $\mathbf{K}_r^{(i)} = (a_1, a_2, ..., a_i - 1, ..., a_r)$ and $w_i = R(\mathbf{K}_r^{(i)}) - 1$.

TABLE I

Known Non-trivial Ramsey numbers

$\mathbf{K}_r$	(3, 3)	(3, 4)	(3, 5)	(3, 6)	(3, 7)	(4, 4)	(3, 3, 3)
$R(\mathbf{K}_r)$	6	9	14	18	23	18	17

The following lemma and corollary are well known. We omit the proofs here since they are also quite easy.

LEMMA 1. *Let G be $\mathbf{K}_r$-colored and G^* any complete subgraph of G. Then "in G^*," there are at most w_i edges of color c_i from any vertex.*

COROLLARY. $R(\mathbf{K}_r) \leqslant 2 + \sum_1^r w_i$, *with strict inequality if some w_i is odd and $\sum_1^r w_i$ is even.*

LEMMA 2. *Let G be a complete graph on $R(\mathbf{K}_r)$ vertices with an edge (v_1, v_2) removed. If G is $\mathbf{K}_r$-colored, then for each $i = 1,..., r$ there exist vertices $v_1^{(i)},..., v_{a_i-2}^{(i)}$ such that all edges between vertices $v_1, v_2, v_1^{(i)},..., v_{a_i-2}^{(i)}$ are colored in color c_i.*

Proof. Since G with edge (v_1, v_2) added is a complete graph on $R(\mathbf{K}_r)$ vertices and cannot be $\mathbf{K}_r$-colored, any coloring of edge (v_1, v_2) with color c_i must introduce a monochromatic complete subgraph on a_i vertices of color c_i. Since G is $\mathbf{K}_r$-colored, the monochromatic complete subgraph must contain edge (v_1, v_2) and hence vertices v_1 and v_2.

II. THE CHROMATIC NUMBER OF A GRAPH G

Let $\bar{G}$ denote the complement of a graph G and K_l a complete graph on l vertices. Let $I_l = \bar{K}_l$ and Q_l denote an l-gon. We write $G = [G_1, G_2,..., G_k]$ if G is the union of disjoint graphs $G_1, G_2,..., G_k$ and $G = (G_1, G_2,..., G_k)$ if G contains a subgraph $[G_1, G_2,..., G_k]$, with every vertex of G in some G_i. As usual, we let $G_1 < G_2$ if G_1 is a subgraph of G_2. Note that $(G_1, G_2,..., G_k)$ differs from $[G_1, G_2,..., G_k]$ by having possibly some more edges. From the definition of the chromatic number of a graph, we have the following lemma:

LEMMA 3. Let G be a given graph. Then $\chi(G)$, the chromatic number of G, is the minimum s such that $\bar{G} = (K_{l_1}, K_{l_2},..., K_{l_s})$.

THEOREM 1. *If* $s = \chi(G) < R(\mathbf{K}_r)$, *then* G *can be* $\mathbf{K}_r$*-colored.*

Proof. Let $\bar{G} = (K_{l_1}, K_{l_2}, ..., K_{l_s})$ and A_i denote the set of vertices in K_{l_i}. Consider a $\mathbf{K}_r$-coloring of K_s with vertices $v_1, v_2, ..., v_s$. We color the edges of

$$G^* = \overline{[K_{l_1}, K_{l_2}, ..., K_{l_s}]}$$

as follows. If $x \in A_i$ and $y \in A_j$, then the edge (x, y) is colored in the same color as edge (v_i, v_j). Clearly, this is a $\mathbf{K}_r$-coloring of G^* because no two vertices in the same A_i have an edge between them in G^*. Since $G < G^*$, the theorem follows.

III. $\mathbf{K}_r$-COLORINGS OF GRAPHS THAT DO NOT CONTAIN A COMPLETE SUBGRAPH ON l VERTICES

Let H_l^v denote the set of all graphs on v vertices that do not contain a K_l subgraph. In this section, we concern ourselves with $\mathbf{K}_r$-coloring of such graphs. Let $N(\mathbf{K}_r, l)$ denote the minimum v such that there exists a graph $G \in H_l^v$ which cannot be $\mathbf{K}_r$-colored. We also obtain some lower bounds for $N(\mathbf{K}_r, l)$. It is clear that $N(\mathbf{K}_r, l) > v$ if and only if all graphs in H_l^v can be $\mathbf{K}_r$-colored.

THEOREM 2. $N(\mathbf{K}_r, R(\mathbf{K}_r) + 1) = R(\mathbf{K}_r)$.

This is immediate from the definition of $R(\mathbf{K}_r)$.

THEOREM 3. $N(\mathbf{K}_r, R(\mathbf{K}_r)) \geqslant R(\mathbf{K}_r) + 2$ *with equality holding if and only if* $G^{\#}$ *cannot be* $\mathbf{K}_r$*-colored, where* $\bar{G}^{\#} = [Q_5, I_{R(\mathbf{K}_r)-3}]$.

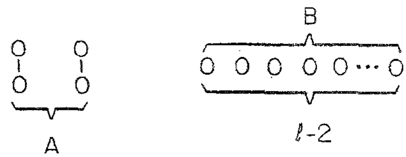
Proof. Let G be any graph $\in H_l^v$ with $v = R(\mathbf{K}_r) + 1$, $l = R(\mathbf{K}_r)$. Then $\bar{G}$ must contain either $[K_2, K_2, I_{R(\mathbf{K}_r)-3}]$ or $[K_3, I_{R(\mathbf{K}_r)-2}]$ as a subgraph. Note that

$$I_t = [\underbrace{K_1, K_1, ..., K_1}_{tK_1's}].$$

Thus $\chi(G) \leqslant R(\mathbf{K}_r) - 1$. By Theorem 1, G can be $\mathbf{K}_r$-colored, and hence $N(\mathbf{K}_r, R(\mathbf{K}_r)) \geqslant R(\mathbf{K}_r) + 2$.

Let G be any graph $\in H_l^v$ with $v = R(\mathbf{K}_r) + 2$ and $l = R(\mathbf{K}_r)$. Since $\bar{G}$

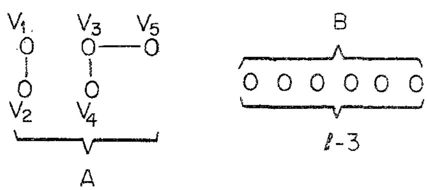
cannot contain a set of l or more independent vertices, $\bar{G}$ must contain the following as a subgraph:



If there is any edge between the vertices in set B, then

$$\bar{G} = (K_2, K_2, K_2, I_{l-4}) \qquad \text{and} \qquad \chi(G) \leqslant l - 1 = R(\mathbf{K}_r) - 1.$$

If there is no edge between the set of vertices A and the set of vertices B and no edge between the vertices in the set B (in $\bar{G}$), then all pairs of vertices in A must be connected by an edge in $\bar{G}$ to prevent a K_l in G. Hence $\bar{G} = (K_4, I_{l-2})$ and again $\chi(G) \leqslant R(\mathbf{K}_r) - 1$. In either case G can be $\mathbf{K}_r$-colored by Theorem 1. Therefore we may assume $\bar{G}$ to contain the following as a subgraph, with no edge between the vertices in set B:



Label the vertices in A as v_1, v_2, v_3, v_4, v_5. If there is an edge between any vertex in B and either vertices v_4 or v_5 in $\bar{G}$, then $\bar{G} = (K_2, K_2, K_2, I_{l-4})$. If edge (v_4, v_5) is in $\bar{G}$, then $\bar{G} = (K_3, K_2, I_{l-3})$. In either case, $\chi(G) \leqslant R(\mathbf{K}_r) - 1$. Hence we may assume $\bar{G}$ does not contain those edges. Now consider the following two sets of l vertices in $\bar{G}$,

$$S_1 = (v_1, v_4, v_5, B), \qquad S_2 = (v_2, v_4, v_5, B).$$

There must be an edge connecting two vertices in each set to prevent a K_l in G. The only possibilities are as follows:

In S_1, edge (v_1, v_4), (v_1, v_5) or (v_1, b).

In S_2, edge (v_2, v_4), (v_2, v_5) or (v_2, b'), $b, b' \in B$.

Of the nine possible choices of one edge each from S_1 and S_2, we see

$$(v_1, v_4) \quad \text{with} \quad (v_2, v_5) \qquad \text{or} \qquad (v_1, v_5) \quad \text{with} \quad (v_2, v_4)$$

gives $\bar{G} = [Q_5, I_{l-3}]$. The other possibilities give either

$$\bar{G} = (K_2, K_2, K_2, I_{l-4})$$

or $\bar{G} = (K_3, K_2, I_{l-3})$ with $\chi(G) \leqslant R(\mathbf{K}_r) - 1$. Hence, if $G \neq G^{\#}$, G can be $\mathbf{K}_r$-colored and $N(\mathbf{K}_r, R(\mathbf{K}_r)) > R(\mathbf{K}_r) + 2$.

THEOREM 4. *Let* $\mathbf{K}_r = (a_1, a_2, ..., a_r)$ *with* $a_i \geqslant 3$, $r \geqslant 2$. *If*

$$R(\mathbf{K}_r) = \sum_1^r [R(\mathbf{K}_r^{(i)}) - 1] + 2 = \left(\sum_1^r w_i\right) + 2,$$

then $G^{\#}$ *cannot be* $\mathbf{K}_r$*-colored, and thus* $N(\mathbf{K}_r, R(\mathbf{K}_r)) = R(\mathbf{K}_r) + 2$.

Proof. Let us denote by Q the set of 5 vertices in Q_5 and I the set of $R(\mathbf{K}_r) - 3$ vertices in $I_{R(\mathbf{K}_r)-3}$. Assume $G^{\#}$ can be $\mathbf{K}_r$-colored. We show

(P) If x is any vertex in I, then the edges from x to all edges in Q must be monochromatic.

Proof. Let d_i, $i = 1, ..., r$ be the number of edges from x to the other vertices in I which are colored in color c_i. Since $G^{\#}$ is $\mathbf{K}_r$-colored,

$$0 \leqslant d_i \leqslant w_i, \qquad \text{for} \quad i = 1, 2, ..., r \quad \text{by Lemma 1.}$$

Since

$$\sum d_i = R(\mathbf{K}_r) - 4 = \left(\sum_1^r w_i\right) - 2$$

we have only two possibilities:

(a) $\exists i, j, \ni d_i = w_i - 1$, $d_j = w_j - 1$ and $d_k = w_k$ for all $k = 1, ..., r$ and $k \neq i$, $k \neq j$,

or

(b) $\exists i, \ni d_i = w_i - 2$, and $d_k = w_k$ for all $k = 1, ..., r$ and $k \neq i$.

Since $d_k = w_k$ implies that the edges from x to Q cannot be colored in color c_k by Lemma 1, (a) implies that the five edges from x to Q can be colored only in colors c_i or c_j. Hence there are at least two vertices q_1, q_2 in Q such that edge $(q_1, q_2) \in G^{\#}$ and edges (x, q_1), (x, q_2) are both colored, say in color c_i. Since $d_i = w_i - 1$, x is then connected to all vertices of the complete subgraph on $w_i + 1 = R(\mathbf{K}_r^{(i)})$ vertices by edges all colored in color c_i. Again, by Lemma 1, this is impossible. Possibility (b) implies, however, that all 5 edges from x to Q are colored in color c_i.

Let us then partition the vertices in I into sets $S_1, S_2, ..., S_r$ according to the color of the edge from that vertex to Q, i.e., $x \in S_i$ if and only if all edges (x, q), $q \in Q$ are colored in color c_i. Let $|S_i|$ be the cardinality of set S_i, then

$$0 \leqslant |S_i| \leqslant w_i, \qquad \text{for} \quad i = 1, 2, \ldots, r \quad \text{by Lemma 1,}$$

and

$$\sum_{i=1}^{r} |S_i| = R(\mathbf{K}_r) - 3 = \sum_{1}^{r} w_i - 1.$$

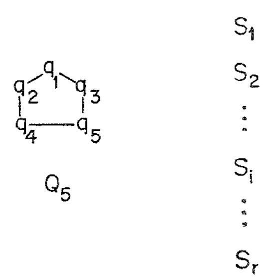
Hence there exists an i such that

$$|S_i| = w_i - 1$$

and

$$|S_j| = w_j, \qquad j = 1, 2, \ldots, r \text{ and } j \neq i.$$

We have now the following configuration:



Since $|S_j| = w_j$ for $j \neq i$, all edges in the pentagon Q_5 must be colored in color c_i. Now consider the subgraph G' formed by the $w_i + 1 = R(\mathbf{K}_r^{(i)})$ vertices q_2, q_3, S_i, all of which are connected to q_1 by edges colored in color c_i. Since $G^{\#}$ is $\mathbf{K}_r$-colored, G' must be $\mathbf{K}_r^{(i)}$-colored. But G' is a complete graph on $R(\mathbf{K}_r^{(i)})$ vertices with edge (q_2, q_3) removed. Hence, by Lemma 2, there exists at least a vertex v in S_i such that edges (v, q_2) and (v, q_3) are colored in color $c_k \neq c_i$. Here we need the assumption that $r \geqslant 2$ and $a_j \geqslant 3$ for $j = 1, 2, \ldots, r$. This is impossible since all vertices in S_i are connected to q_2 and q_3 by edges colored in color c_i. Hence $G^{\#}$ cannot be $\mathbf{K}_r$-colored and Theorem 4 is proved.

Among the known Ramsey numbers, $R(3, 3)$, $R(3, 5)$, $R(4, 4)$, and $R(3, 3, 3)$ all satisfy the hypothesis of Theorem 4. Therefore, we have the following corollary:

$$N((3, 3), 6) = 8,$$
$$N((3, 5), 14) = 16,$$
$$N((4, 4), 18) = 20,$$
$$N((3, 3, 3), 17) = 19.$$

The unique graphs $G^{\#}$ which give the above equality are given by $\bar{G}^{\#} = [Q_5, I_3]$, $[Q_5, I_{11}]$, $[Q_5, I_{15}]$, and $[Q_5, I_{14}]$, respectively.

The question whether $N((3, 3), 6)$ exists was first asked by Erdös and Hajnal [7]. J. H. van Lint (unpublished) first showed that $N((3, 3), 6) \leqslant 14$ and R. L. Graham in [8] showed that $N((3, 3), 6) = 8$ by producing the graph

$$G^{\#} = \overline{[Q_5, I_3]}.$$

To show that the assumption

$$R(\mathbf{K}_r) = \sum_1^r [\mathbf{R}(K_r^{(i)}) - 1] + 2$$

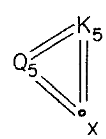
is essential in Theorem 4, we give the following example in Theorem 5.

For $\mathbf{K}_r = (3, 4)$, $R(\mathbf{K}_r) = 9 \leqslant R(3, 3) + R(2, 4) = 10$. Let $\bar{G}^{\#} = [Q_5, I_6]$.

THEOREM 5. *$G^{\#}$ can be (3, 4)-colored. Hence*

$$N((3, 4), 9) > 11.$$

Proof. We represent $G^{\#}$ as follows



where $A = B$ indicates that all vertices from A are joined to all vertices in B. Let c_1 be black and c_2 be red. We will color the edges of $G^{\#}$ such that there are no black triangles and no red complete quadrilaterals as follows:

(1) all edges in Q_5 red,

(2) x to all vertices in Q_5 black,

(3) x to all vertices in K_5 red.

Let the vertices in Q_5 be labeled q_1, q_2, q_3, q_4, q_5 with red edges (q_1, q_2), (q_2, q_3), (q_3, q_4), (q_4, q_5), (q_5, q_1) and the vertices in K_5 be labeled k_1, k_2, k_3, k_4, k_5 with black edges (k_1, k_2), (k_2, k_3), (k_3, k_4), (k_4, k_5), (k_5, k_1) and red edges (k_1, k_3), (k_2, k_4), (k_3, k_5), (k_4, k_1), (k_5, k_2).

The colorings of the edges connecting Q_5 and K_5 are given in Table II. The complete (3, 4)-coloring of $G^{\#}$ is given in Fig. 1.

It is a trivial matter to check that there are no black triangles. Since K_5 does not contain a red triangle, no red complete quadrilateral can involve x, and consequently any possible red complete quadrilateral must have

TABLE II

	k_1	k_2	k_3	k_4	k_5
q_1	r	b	r	r	b
q_2	b	r	b	r	r
q_3	r	b	r	b	r
q_4	r	r	b	r	b
q_5	b	r	r	b	r

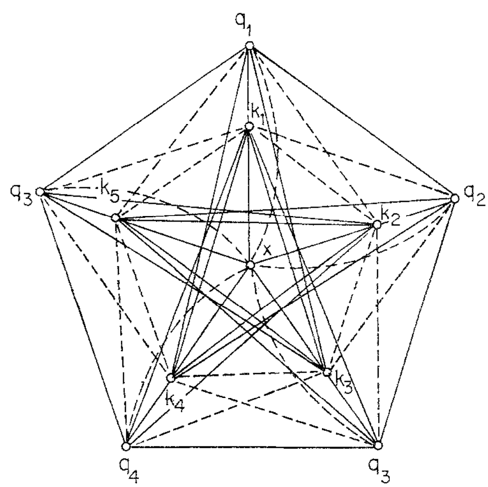


FIG. 1. (3, 4)-coloring of $\overline{[Q_5, I_6]}$: (– – –) black, (——) red.

two connected vertices in Q_5 and two others in K_5. An inspection of Table II shows that the 5 pairs of rows (q_1, q_2), (q_2, q_3), (q_3, q_4), (q_4, q_5), (q_5, q_1) all have just one red entry in a common column. Hence there are no red complete quadrilaterals and Theorem 5 is proved.

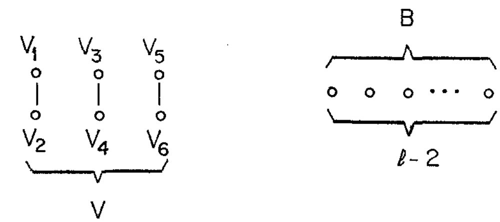
IV. A Lower Bound for $N(\mathbf{K}_r, R(K_r) - 1)$

In [9], R. L. Graham and J. H. Spencer showed that $N((3, 3), 5) \leqslant 23$ by producing a graph $G \in H_5^{23}$ which cannot be (3, 3)-colored. They raised the question whether $N((3, 3), 5) \geqslant 10$. In the following, we show that $N((3, 3), 5) \geqslant 10$ by proving the more general Theorem 6:

THEOREM 6. $N(\mathbf{K}_r, R(\mathbf{K}_r) - 1) \geqslant R(\mathbf{K}_r) + 4$.

Proof. Let $G \in H_l^v$ where $v = R(\mathbf{K}_r) + 3$ and $l = R(\mathbf{K}_r) - 1$. We show that $\chi(G) \leqslant l = R(\mathbf{K}_r) - 1$ and hence G can be $\mathbf{K}_r$-colored. This in turn means that $N(\mathbf{K}_r, R(\mathbf{K}_r) - 1) \geqslant R(\mathbf{K}_r) + 4$.

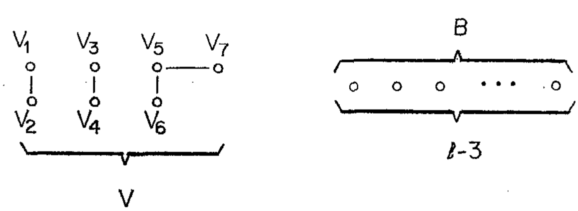
Since G does not contain a K_l subgraph, we may assume $\bar{G}$ to contain the following subgraph:



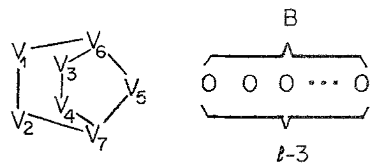
(1) If there are any more edges between two vertices in B, then $\bar{G} = (K_2, K_2, K_2, K_2, I_{l-4})$ and $\chi(G) \leqslant l$.

(2) If there are no edges joining the set of vertices in V to B and no edges between two vertices in B, then every pair of vertices in V must be connected by an edge in $\bar{G}$ to prevent a K_l in G. But then $\bar{G} = (K_6, I_{l-2})$ and $\chi(G) \leqslant l - 1$.

Therefore, we may assume $\bar{G}$ to contain the following subgraph:



with no edges between the vertices in B, and no edges from either v_6 or v_7 to B. Furthermore, if $\bar{G}$ contains the edge (v_6, v_7), then $\bar{G} = (K_3, K_2, K_2, I_{l-3})$ and $\chi(G) \leqslant l$. Hence (v_6, v_7, B) form a set of $l - 1$ vertices in $\bar{G}$ with no edges. To prevent a K_l in G, v_1, v_2, v_3, v_4 must each be joined to some vertex in the set (v_6, v_7, B) by an edge. Suppose v_1 is joined to some $b \in B$. Then any edge from v_2 to (v_6, v_7, B) produces either $\bar{G} = (K_2, K_2, K_2, K_2, I_{l-4})$ or $\bar{G} = (K_3, K_2, K_2, I_{l-3})$; the latter case arises when v_1 and v_2 are both joined to the same $b \in B$. Suppose v_1 and v_2 are both joined to either v_6 or v_7, then $\bar{G} = (K_3, K_2, K_2, I_{l-3})$. Similar arguments hold for v_3 and v_4 and we must have $\bar{G}$ containing the following subgraph:



with no edge from (v_1, v_2, v_3, v_4) to B.

Consider now the two sets of l vertices (v_1, v_3, v_5, B) and (v_2, v_4, v_5, B) in $\bar{G}$. There must be an edge joining some two vertices in each set to prevent a K_l in G. In view of the above arguments, the only possibilities left are: an edge (v_5, b), $b \in B$, or one edge from each of the triangles (v_1, v_3, v_5) and (v_2, v_4, v_5). The former gives $\bar{G} = (K_2, K_2, K_2, K_2, I_{l-4})$ while the rest of the possibilities give either $\bar{G} = (K_3, K_2, I_{l-3})$ or $\bar{G} = (K_3, K_3, I_{l-2})$. Hence we have proved that $G \in H_l^v$ implies $\chi(G) \leqslant l = R(\mathbf{K}_r) - 1$ and hence G can be $\mathbf{K}_r$-colored by Theorem 1. Hence $N(\mathbf{K}_r, R(\mathbf{K}_r) - 1) \geqslant R(\mathbf{K}_r) + 4$.

COROLLARY:

$$N((3, 3), 5) \geqslant 10$$
$$N((3, 4), 8) \geqslant 13$$
$$N((3, 5), 13) \geqslant 18$$
$$N((3, 6), 17) \geqslant 22$$
$$N((3, 7), 22) \geqslant 27$$
$$N((4, 4), 17) \geqslant 22$$
$$N((3, 3, 3), 16) \geqslant 21$$

Admittedly, the bounds given by Theorem 6 are still quite weak. One can presumably show that $N(\mathbf{K}_r, R(\mathbf{K}_r) - 1) \geqslant R(\mathbf{K}_r) + 5$ by using essentially the same procedures, except that the work may be too lengthy to present. However, since the results obtained so far for the existence of $N((3, 3), 4)$ by J. H. Folkman [10] seem to indicate that $N((3, 3), 4)$ is enormous, any reasonable bound for $N(\mathbf{K}_r, R(\mathbf{K}_r) - 2)$ in general may be very difficult.

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