SIAM J. APPL. MATH.
Vol. 18, No. 1, January 1970

GRAPHS WITH MONOCHROMATIC COMPLETE SUBGRAPHS IN EVERY EDGE COLORING*

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Abstract. For integers $r, s \geqq 2$, let $\Gamma(r, s)$ be the class of all graphs G with the following property: if the edges of G are colored red and blue, then either G contains r mutually adjacent vertices with all connecting lines colored red, or s mutually adjacent vertices with all connecting lines colored blue. By Ramsey's theorem, $\Gamma(r, s)$ contains all sufficiently large complete graphs. It follows that $\Gamma(r, s)$ contains all graphs with a sufficiently large number of mutually adjacent vertices. We are concerned here with determining the minimum number $f = f(r, s)$ such that $\Gamma(r, s)$ contains a graph G with at most f mutually adjacent vertices. Obviously $f(r, s) \geqq \max(r, s)$. We show constructively that $f(r, s) = \max(r, s)$.[1]

1. Introduction. By a graph we mean a finite undirected graph with no edge joining a vertex to itself and at most one edge joining any pair of distinct vertices. By an isomorphism from a graph G to a graph H we mean a one-to-one mapping φ from the vertices of G onto the vertices of H with the property that two vertices v and v' of G are adjacent in G if and only if $\varphi(v)$ and $\varphi(v')$ are adjacent in H. If S is a subset of the vertices of a graph G, the subgraph of G *spanned* by S is the graph whose vertices are the elements of S and whose edges are *all* edges of G joining elements of S. If S is any set, $|S|$ will denote the cardinality of S.

We define $\delta(G)$, the *dimension* of a graph G, by setting $\delta(G)$ equal to the largest integer δ such that G contains δ mutually adjacent vertices. The term "dimension" is suggested by the observation that if G is regarded as a 1-dimensional simplicial complex, then $\delta(G) - 1$ is the dimension of the largest simplicial complex having G as its 1-dimensional skeleton.

For integers $k_1, k_2 \geqq 2$, let $\Gamma(k_1, k_2)$ denote the class of all graphs G with the following property: if the edges of G are partitioned into classes C_1 and C_2, then for some $i = 1$ or 2 there are k_i mutually adjacent vertices of G with all the edges joining them in the class C_i. By Ramsey's theorem there is an integer $N = N(k_1, k_2)$ such that K_N, the complete graph on N vertices, is in $\Gamma(k_1, k_2)$. Let $f(k_1, k_2) = \min\{\delta(G) | G \in \Gamma(k_1, k_2)\}$. Our purpose here is to compute the function f. This investigation was motivated by the question[2] (first raised by P. Erdös for the case $k_1 = k_2 = 3$) of whether or not $f(k_1, k_2) = N(k_1, k_2)$. We show that except in the trivial case $k_1 = 2$ or $k_2 = 2$ equality does not hold. In fact we have the following result.

* Received by the editors November 30, 1967. Presented by invitation at the Symposium on Combinatorial Mathematics, sponsored by the Office of Naval Research, at the 1967 Fall Meeting of Society for Industrial and Applied Mathematics held at the University of California, Santa Barbara, California, November 29–December 2, 1967.

Due to the author's tragic and untimely decease, this posthumous paper is being published as first submitted. However, some editorial footnotes based on the author's oral presentation have been added and the references have been supplied.

[1] *Editor's note.* For example, if $r = s = 3$, the construction yields a very large graph which contains no complete graph on 4 points, yet has the property that any coloring of the edges must yield either a red triangle or a blue triangle.

[2] *Editor's note.* A special case of the problem solved in this paper was stated in Erdös and Hajnal [1] and solved in Graham [2].

THEOREM 1. $f(k_1, k_2) = \max(k_1, k_2)$.

The proof of Theorem 1 relies heavily on the following result, which is of interest in its own right.

THEOREM 2. *For each positive integer n and each graph G there is a graph $H(n, G)$ with the following properties:*

(a) $\delta(H(n, G)) = \delta(G)$;

(b) *if the vertices of $H(n, G)$ are partitioned into classes $C_1, \cdots, C_n$, then for some i, $1 \leqq i \leqq n$, there is a set $S \subseteq C_i$ such that the subgraph of $H(n, G)$ spanned by S is isomorphic to G.*

2. Proofs of the theorems.

2.1. Proof of Theorem 2.[3] For $n = 1$ we set $H(1, G) = G$. We construct $H(2, G)$ by induction on the number of vertices of G. If G has only one vertex, we may take $H(2, G) = G$. Now suppose that G has $r \geqq 2$ vertices and suppose that $H(2, G')$ has been defined for every graph G' with fewer than r vertices. Let V be the set of vertices of G. Let $v_0 \in V$. Let $V' = V - \{v_0\}$ and let V'' be the set of vertices of G which are adjacent to v_0. Let G' and G'' be the subgraphs of G spanned by the vertices in V' and V'', respectively.

By the inductive assumption, $H(2, G')$ has been defined.[4] Let W be the set of vertices of $H(2, G')$. Let X be the collection of all subsets of W that span a subgraph of $H(2, G')$ which is isomorphic to G''. Condition (b) applied to $H(2, G')$ and the definition of G' and G'' imply that X is nonempty. Let $I = \{1, 2, \cdots, 2^{|W|}r\}$ and let J be the collection of r-element subsets of I. For each set $T \in J$ let f_T be a one-to-one function from T onto V.

We define a graph H as follows: the vertices of H are the elements of the set $(V \times X \times J) \cup (W \times I)$. Let (v, S, T), $(v', S', T') \in V \times X \times J$ and let (w, i), $(w', i') \in W \times I$. Then (v, S, T) and (v', S', T') are adjacent in H if and only if $S = S'$, $T = T'$ and v is adjacent to v' in G. (v, S, T) and (w, i) are adjacent if and only if $w \in S$, $i \in T$ and $f_T(i) = v$. (w, i) and (w', i') are adjacent if and only if $i = i'$ and w is adjacent to w' in $H(2, G')$.

Suppose that the vertices of H are partitioned into two classes C_1 and C_2. For $k = 1$ or 2 and $i \in I$, let $C_{k,i} = \{w \in W | (w, i) \in C_k\}$. For each $i \in I$, $(C_{1,i}, C_{2,i})$ is a partition of W. There are only $2^{|W|}$ such partitions of W and $|I| = 2^{|W|}r$. Consequently, there is a partition (D_1, D_2) of W and a set $T \subset I$ with $|T| = r$ such that $C_{k,i} = D_k$ for $k = 1$ or 2 and $i \in T$. By condition (b) on $H(2, G')$, there is a $k = 1$ or 2 and a set $S' \subset D_k$ such that the subgraph of $H(2, G')$ spanned by S' is isomorphic to G'. Let $\varphi : V' \to S'$ be the isomorphism. Let $S'' = \varphi(V'')$. Then φ restricted to V'' is an isomorphism of G'' with the subgraph of $H(2, G')$ spanned by S''. Hence, $S'' \in X$. Suppose that $(v_1, S'', T) \in C_k$ for some $v_1 \in V$. Then $E = (S' \times \{f_T^{-1}(v_1)\}) \cup \{(v_1, S'', T)\}$ is a subset of C_k.

Define $\bar{\varphi} : V \to E$ by

$$\bar{\varphi}(v) = \begin{cases} (v_1, S'', T) & \text{if } v = v_0, \\ (\varphi(v), f_T^{-1}(v_1)) & \text{if } v \neq v_0. \end{cases}$$

[3] *Editor's note.* The main portion of this proof is devoted to $H(2, G)$. Once this portion is completed, it is easy to handle $H(n, G)$.

[4] *Editor's note.* Intuitively speaking, $H(2, G)$ is constructed by taking the union of many copies of G and many copies of $H(2, G')$, and adding certain edges from certain points in copies of G to certain points in copies of $H(2, G')$.

It is easily verified that $\bar{\varphi}$ is an isomorphism from G to the subgraph of H spanned by E. On the other hand, if $(v, S'', T) \notin C_k$ for all $v \in V$, then $\{(v, S'', T) | \text{ all } v \text{ in } V\} \subset C_l$ where $l \in \{1, 2\} - \{k\}$. The function $\psi: V \to V \times \{S''\} \times \{T\}$ defined by $\psi(v) = (v, S'', T)$ is an isomorphism of G with the subgraph of H spanned by $V \times \{S''\} \times \{T\}$. Hence, in either case we see that H satisfies condition (b) required of the graph $H(2, G)$.

Now we prove that H has the same dimension as G. Let $\delta = \delta(H)$ and let A be a set of mutually adjacent vertices of H with $|A| = \delta$. We consider three cases. First, suppose that A contains at least two vertices from the set $V \times X \times J$. Since these vertices are adjacent, they must be of the form (v_1, S, T) and (v_2, S, T), where $S \in X$, $T \in J$ and v_1, v_2 are adjacent vertices of G. If $(w, i) \in W \times I$ and (w, i) is adjacent to (v_1, S, T), then $f_T(i) = v_1 \neq v_2$ so (w, i) is not adjacent to (v_2, S, T). Hence, $A \subset V \times X \times J$ so A has the form $A = \{(v_1, S, T), \cdots, (v_\delta, S, T)\}$, where $v_1, \cdots, v_\delta$ are mutually adjacent vertices of G. Therefore, $\delta \leqq \delta(G)$. Now suppose that A contains exactly one vertex, (v, S, T), in $V \times X \times J$. Then $A = \{(v, S, T), (w_1, f_T^{-1}(v)), \cdots, (w_{\delta-1}, f_T^{-1}(v))\}$, where $w_1, \cdots, w_{\delta-1}$ are mutually adjacent vertices of $H(2, G')$ which all lie in S. Now the set $B = \{(v, S, T)\} \cup (S \times \{f_T^{-1}(v)\})$ spans a subgraph of H isomorphic to the subgraph of G spanned by the set $\{v_0\} \cup V''$. Since $A \subset B$, this implies that $\delta \leqq \delta(G)$. Finally, suppose that A contains no vertex of $V \times X \times J$. Then $A = \{(w_1, i), \cdots, (w_\delta, i)\}$, where $i \in I$ and $w_1, \cdots, w_\delta$ are mutually adjacent vertices of $H(2, G')$. Hence, $\delta \leqq \delta(H(2, G')) = \delta(G') \leqq \delta(G)$. We have now shown that $\delta(H) \leqq \delta(G)$. By condition (b), H contains a subgraph isomorphic to G, so $\delta(H) = \delta(G)$. Hence, we may set $H(2, G) = H$. By induction, $H(2, G)$ is defined for all graphs G.

Now let $n > 2$ and suppose that $H(n - 1, G)$ is defined for all graphs G. Set $H(n, G) = H(2, H(n - 1, G))$. We have $\delta(H(2, H(n - 1, G))) = \delta(H(n - 1, G)) = \delta(G)$ so (a) is satisfied. Let $C_1, \cdots, C_n$ be a partitioning of the vertices of $H(2, H(n - 1, G))$ into n classes. Let $D_1 = C_1 \cup \cdots \cup C_{n-1}$ and $D_2 = C_n$. For some $i = 1$ or 2, there is a set $S \subset D_i$ that spans a subgraph H of $H(2, H(n - 1, G))$ such that H is isomorphic to $H(n - 1, G)$. If $i = 1$, then $S \cap C_1, \cdots, S \cap C_{n-1}$ is a partitioning of S into $n - 1$ classes. If $i = 2$, then $S \cap C_n, \varnothing, \cdots, \varnothing$ is a partitioning of S into $n - 1$ classes. In either case there is a set $S' \subset S \cap C_j$ for some j with $1 \leqq j \leqq n$ such that H', the subgraph of H spanned by S', is isomorphic to G. Since H' is also the subgraph of $H(2, H(n - 1, G))$ spanned by S', it follows that condition (b) is satisfied.

2.2. Proof of Theorem 1. Let $k_1, k_2 \geqq 2$ and let $G \in \Gamma(k_1, k_2)$. Let (C_1, C_2) be the partitioning of the edges of G that places all of the edges in the class C_1. For some $i = 1$ or 2 there are k_i mutually adjacent vertices of G with all edges joining them in the class C_i. Now $k_i \geqq 2$ so C_i contains at least one edge. Hence, $i = 1$ so $\delta(G) \geqq k_1$. Similarly, $\delta(G) \geqq k_2$. Since G was an arbitrary member of $\Gamma(k_1, k_2)$, $f(k_1, k_2) \geqq \max(k_1, k_2)$.

Now let $\{i, j\} = \{1, 2\}$ and suppose that $k_i = 2$. Then K_{k_j}, the complete graph on k_j vertices, is in $\Gamma(k_1, k_2)$. Hence, $\max(k_1, k_2) \leqq f(k_1, k_2) \leqq \delta(K_{k_j}) = k_j = \max(k_1, k_2)$.

To prove $f(k_1, k_2) = \max(k_1, k_2)$ in the general case, we use induction on $k_1 + k_2$. If $k_1 + k_2 \leqq 5$, then $k_i = 2$ for $i = 1$ or 2 and the equality has already been established. Suppose that $k_1 + k_2 \geqq 6$ and that $f(k_1', k_2') = \max(k_1', k_2')$ whenever $k_1', k_2' \geqq 2$ and $k_1' + k_2' < k_1 + k_2$. If $k_1 = 2$ or $k_2 = 2$ we have already established the equality, so we suppose that $k_1, k_2 \geqq 3$. Let $m = \max(k_1, k_2)$. By the inductive assumption there are graphs $G_1 \in \Gamma(k_1 - 1, k_2)$, $G_2 \in \Gamma(k_1, k_2 - 1)$ and $G_3 \in \Gamma(k_1 - 1, k_2 - 1)$ with $\delta(G_1) = \max(k_1 - 1, k_2) \leqq m$, $\delta(G_2) = \max(k_1, k_2 - 1) \leqq m$ and $\delta(G_3) = \max(k_1 - 1, k_2 - 1) = m - 1$. To complete the proof of Theorem 1, it suffices to construct a graph $G \in \Gamma(k_1, k_2)$ with $\delta(G) \leqq m$.

We may assume that G_1 and G_2 are chosen so that they have no edge or vertex in common. Then $G_1 \cup G_2$ is a well-defined graph and $\delta(G_1 \cup G_2) = \max(\delta(G_1), \delta(G_2)) \leqq m$. Let M be the number of $m - 1$ element subsets of the vertices of $G_1 \cup G_2$. Let N be the number of ways the edges of $H((m-1)^2M^2, G_3)$ can be partitioned into two classes. (Here $H(n, G)$ is as in Theorem 2.) Let V_1 be the set of vertices of $H(N, G_1 \cup G_2)$ and let V_2 be the set of vertices of $H((m-1)^2M^2, G_3)$. Finally, let X be the collection of all sets $S \subset V_1$ with $|S| = m - 1$.

We construct a graph G as follows:[5] The vertices of G are the elements of the set $(V_1 \times V_2) \cup X$. If $(v_1, v_2), (v_1', v_2') \in V_1 \times V_2$, then (v_1, v_2) and (v_1', v_2') are adjacent in G if and only if $v_1 = v_1'$ and v_2 is adjacent to v_2' in $H((m-1)^2M^2, G_3)$ or $v_2 = v_2'$ and v_1 is adjacent to v_1' in $H(N, G_1 \cup G_2)$. If $(v_1, v_2) \in V_1 \times V_2$ and $S \in X$, then (v_1, v_2) and S are adjacent in G if and only if $v_1 \in S$. If $S, S' \in X$ then S and S' are not adjacent in G.

We first show that $\delta(G) \leqq m$. Let A be a set of mutually adjacent vertices of G with $|A| = \delta = \delta(G)$. Suppose that A contains an element of X. Then A contains exactly one element of X so $A = \{S, (v_1, v_1'), \cdots, (v_{\delta-1}, v_{\delta-1}')\}$, where $S \in X$, $v_i \in S \subset V_1$ and $v_i' \in V_2$ for $1 \leqq i \leqq \delta - 1$. If $v_1, \cdots, v_{\delta-1}$ are distinct, then $\delta - 1 \leqq |S| = m - 1$ so $\delta \leqq m$. If these vertices are not distinct, we may assume that $v_1 = v_2$. Then v_1' and v_2' are adjacent in $H((m-1)^2M^2, G_3)$ so $v_1' \neq v_2'$. For $3 \leqq i \leqq \delta - 1$, either $v_i' \neq v_1'$ or $v_i' \neq v_2'$ so either $v_i = v_1$ or $v_i = v_2$. It follows that $v_1 = v_2 = \cdots = v_{\delta-1}$ and $v_1', v_2', \cdots, v_{\delta-1}'$ are distinct mutually adjacent vertices of $H((m-1)^2M^2, G_3)$. Hence, $\delta - 1 \leqq \delta(H((m-1)^2M^2, G_3)) = \delta(G_3) = m - 1$ so $\delta \leqq m$.

Now suppose that A contains no element of X. Then $A = \{(v_1, v_1'), (v_2, v_2'), \cdots, (v_\delta, v_\delta')\}$ where $v_i \in V_1$ and $v_i' \in V_2$ for $1 \leqq i \leqq \delta$. Since (v_1, v_1') and (v_2, v_2') are adjacent in G, either $v_1 = v_2$ or $v_1' = v_2'$. Reasoning as above we see that either $v_1 = v_2 = \cdots = v_\delta$ and $v_1', v_2', \cdots, v_\delta'$ are distinct mutually adjacent vertices of $H((m-1)^2M^2, G_3)$, or $v_1' = v_2' = \cdots = v_\delta'$ and $v_1, v_2, \cdots, v_\delta$ are distinct mutually adjacent vertices of $H(N, G_1 \cup G_2)$. In the first case, $\delta \leqq \delta(H((m-1)^2M^2, G_3)) = \delta(G_3) = m - 1 < m$. In the second case, $\delta \leqq \delta(H(N, G_1 \cup G_2)) = \delta(G_1 \cup G_2) \leqq m$.

It remains to show that $G \in \Gamma(k_1, k_2)$. Let (C_1, C_2) be a partition of the edges of G into two classes. Let $u \in V_1$. If $v_1, v_2 \in V_2$ are adjacent in $H((m-1)^2M^2, G_3)$, then $\{(u, v_1), (u, v_2)\}$ is an edge of G. For $i = 1$ or 2 let $D_i(u)$ be the set of edges, $\{v_1, v_2\}$, of $H((m-1)^2M^2, G_3)$ such that $\{(u, v_1), (u, v_2)\} \in C_i$. Then $(D_1(u), D_2(u))$ is a partition of the edges of $H((m-1)^2M^2, G_3)$. There are exactly N such partitions, so we may partition the vertices of $H(N, G_1 \cup G_2)$ into N classes by putting u and u' into the

[5] *Editor's note.* Intuitively speaking, the construction starts with the Cartesian product of the graphs $H(N, G_1 \cup G_2)$ and $H((m-1)^2M^2, G_3)$. The elements of X are adjoined as new vertices and certain edges are added between new and old vertices.

same class if and only if $(D_1(u), D_2(u)) = (D_1(u'), D_2(u'))$. By Theorem 2, there is a set $U \subset V_1$ which spans a subgraph of $H(N, G_1 \cup G_2)$ isomorphic to $G_1 \cup G_2$ and with the additional property that, for $u \in U$, $(D_1(u), D_2(u)) = (D_1, D_2)$ where (D_1, D_2) is some fixed partition of the edges of $H((m-1)^2 M^2, G_3)$.

Let $v \in V_2$. Then $U \times \{v\}$ spans a subgraph of G isomorphic to $G_1 \cup G_2$. If, for some $v \in V_2$ and some $i = 1$ or 2, this subgraph contains k_i mutually adjacent vertices with all edges connecting them in the class C_i, then we have finished. Otherwise, from the choice of G_1 and G_2 it follows that for each $v \in V_2$ there are sets $S_1(v), S_2(v) \subset U$ such that, for $i = 1$ or 2, $|S_i(v)| = k_i - 1$, the set $S_i(v) \times \{v\}$ of vertices of G are mutually adjacent in G and all edges joining them are in C_i. For $v \in V_2$ and $i = 1$ or 2, choose $T_i(v)$ so $S_i(v) \subset T_i(v) \subset U$ and $|T_i(v)| = m - 1$. Now $U \subset V_1$ so $T_i(v) \in X$ is a vertex of G which is adjacent to every vertex in the set $T_i(v) \times V_2 \supset T_i(v) \times \{v\}$. If, for some $v \in V_2$ and some $i = 1$ or 2, all of the edges of G joining $T_i(v)$ to vertices in $T_i(v) \times \{v\}$ are in the class C_i, then $\{T_i(v)\} \cup (S_i(v) \times \{v\})$ is a set of k_i mutually adjacent vertices of G with all connecting edges in C_i, and we have finished. Otherwise, for each $v \in V_2$ there are vertices $u_1(v) \in T_1(v)$ and $u_2(v) \in T_2(v)$ such that the edge $\{T_1(v), (u_1(v), v)\}$ is in C_2 and the edge $\{T_2(v), (u_2(v), v)\}$ is in C_1.

Now U has the same cardinality as the set of vertices of $G_1 \cup G_2$ so there are exactly M subsets of U with $m - 1$ elements. Hence, there are $(m-1)^2 M^2$ ordered quadruples (u_1, u_2, T_1, T_2) with $u_1 \in T_1 \subset U, u_2 \in T_2 \subset U$ and $|T_1| = |T_2| = m - 1$. We partition the vertices of $H((m-1)^2 M^2, G_3)$ into $(m-1)^2 M^2$ classes by putting v and v' in the same class if and only if $(u_1(v), u_2(v), T_1(v), T_2(v)) = (u_1(v'), u_2(v'), T_1(v'), T_2(v'))$. By Theorem 2, there is a set $V \subset V_2$ such that V spans a subgraph of $H((m-1)^2 M^2, G_3)$ isomorphic to G_3 and such that $(u_1(v), u_2(v), T_1(v), T_2(v)) = (u_1, u_2, T_1, T_2)$ is independent of v as v ranges over V. We identify G_3 with the subgraph spanned by V.

Since $u_1, u_2 \in U$, we have $(D_1(u_1), D_2(u_1)) = (D_1(u_2), D_2(u_2)) = (D_1, D_2)$. Since $G_3 \in \Gamma(k_1 - 1, k_2 - 1)$ and (D_1, D_2) is a partition of the edges of G_3, for some $i = 1$ or 2 there is a set $W \subset V$ such that $|W| = k_i - 1$, the vertices of G_3 that are in W are mutually adjacent and all edges connecting them are in D_i. Let $j \in \{1, 2\} - \{i\}$. Then $\{u_j\} \times W$ is a set of $k_i - 1$ mutually adjacent vertices of G. Since $D_i = D_i(u_j)$, it follows from the definition of $D_i(u_j)$ that all edges joining elements of $\{u_j\} \times W$ are in C_i. Finally, for $w \in W \subset V$, $(u_j, w) = (u_j(w), w)$ is adjacent to $T_j(w) = T_j$ and the edge joining them is in C_i. Hence, $(\{u_j\} \times W) \cup \{T_j\}$ is a set of k_i mutually adjacent vertices of G with all interconnecting edges in C_i.

3. Remarks. If n is any positive integer and $k_i \geqq 2$ is an integer for $1 \leqq i \leqq n$, we may define $\Gamma(k_1, \cdots, k_n)$ to be the class of all graphs G with the following property: if the edges of G are partitioned into classes $C_1, \cdots, C_n$, then for some i, $1 \leqq i \leqq n$, there are k_i mutually adjacent vertices in G such that all the edges joining them are in C_i. Again by Ramsey's theorem, $\Gamma(k_1, \cdots, k_n)$ contains all sufficiently large complete graphs. We set

$$f(k_1, \cdots, k_n) = \min \{\delta(G) | G \in \Gamma(k_1, \cdots, k_n)\}.$$

I conjecture that

$$f(k_1, \cdots, k_n) = \max (k_1, \cdots, k_n)$$

for arbitrary n; however, the methods used here do not seem to be extendable to the case $n > 2$.

A straightforward generalization of the proof of Theorem 1 yields the following inequality: if $k_1 \geqq k_2 \geqq \cdots \geqq k_n \geqq 2$, then

$$k_1 \leqq f(k_1, \cdots, k_n) \leqq k_1 + \min\left(\frac{1}{2}\sum_{i=2}^{n}(k_i - 2), \sum_{i=3}^{n}(k_i - 2)\right).$$

For $n \geqq 3$, I feel that this upper bound is somewhat spurious in the sense that it depends much more on the particular construction used to prove it than it does on the function f.

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[2] R. L. Graham, *On edgewise 2-colored graphs with monochromatic triangles and containing no complete hexagon*, Ibid., 4 (1968), p. 300.