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EUCLIDEAN RAMSEY THEOREMS, II

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1. INTRODUCTION

We first recall a few definitions and results from [2]. Let K be a set of points in Euclidean n-space, E^n, and let the points of E^n be r-colored (that is, divided into r classes, or colors). Then if all the points of K are the same color (i.e., in the same class), K is said to be *monochromatic.* Let H be a group of transformations on E^n. In [2] we were primarily concerned with whether or not the following statement is true: $R_H(K, n, r)$: For any r-coloring of the points of E^n there is a monochromatic set K' which is the image of K under some element of H.

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In this paper, we generalize $R_H(K, n, r)$ in several ways and try to decide which statements are true and which are false. The generalizations include replacing E^n by infinite dimensional spaces, $E^{\aleph_0}$ and $\mathscr{H}$, $\aleph_0$-dimensional Euclidean space and Hilbert space, respectively. We also generalize by replacing K by r different sets K_i and see if for some i there is a K_i', congruent to K_i, which is monochromatic in the i-th color. These latter questions are dealt with in the next section.

If K is a set in some E^m, we say that K is *Ramsey* if for every r there is an n such that $R_H(K, n, r)$ holds, where H is the group of congruences of E^n. We recall that in [2] we showed that a set K is Ramsey if it is a subset of vertices of a "brick", or rectangular parallelepiped, and that if K is Ramsey, it must be a set lying on some sphere. For infinite dimensional spaces the analogues of these may not hold (see Section 4 below).

We conclude this section with a question more properly belonging to the first part of this study [2]. For every n, r is there a number $m = m(n, r)$ such that for every set K of m points there is an r-coloring of E^n without a monochromatic copy of K?

2. ASYMMETRIC RAMSEY PROBLEMS

In this section we consider asymmetric generalizations of the statement $R_H(K, n, r)$, where we write just $R(K, n, r)$ when it is clear that the group H is the group of congruences. The generalization is as follows.

$R_H(K_1, \ldots, K_r, n, r)$: For any r-coloring of E^n there is some i and some K_i' consisting only of points of the i-th color such that K_i' is the image of K_i under some element of H.

Frequently such asymmetric questions can be answered by direct application of a corresponding symmetric theorem. For instance, in the case of finite sets, if we 2-color the k-subsets of an n-set for n sufficiently large, then there is either an l_1-set with all its k-subsets color 1, or an l_2-set with all its k-subsets color 2. This is a direct result of Ramsey's theorem (Theorem 1 of [2]), since we get a monochromatic l-set, where $l = \max(l_1, l_2)$. Ramsey's theorem itself is sometimes stated in this equivalent asymmetric form [6]. It is possible that obtaining bounds on the number n required for a given set of parameters (e.g. l_1, l_2, k) is easier in some cases than obtaining bounds for corresponding symmetric cases [1]. Hopefully some of the geometrical asymmetric theorems discussed here will help provide proofs for yet undecided symmetric cases.

Theorem 1. *If the points of* E^3 *are 2-colored (red and blue), then either there is a red pair of points distance* 1 *apart, or there are four blue points in a line with distance* 1 *separating adjacent points.*

Proof. By Theorem 8 of [2], we know that for any 2-coloring of E^3 there is a monochromatic triple a, b, c of points in a line distance 1 apart. Suppose E^3 is colored to avoid both a red pair distance 1 apart, and a blue set of four points on a line 1 apart. Then we must get a blue triple a, b, c (See Figure 1).

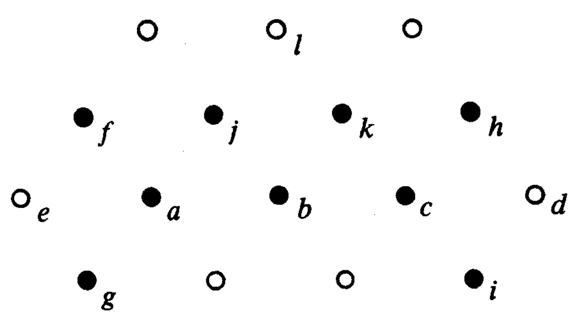

Figure 1

In Figure 1 we have a part of the triangular lattice, with the unit cell an equilateral triangle of side length 1. Then both points d and e must be red, or else we would have four blue points in a row. Thus f, g, h and i must all be blue, as they are adjacent to red points. Now, not both j and k can be red, so suppose j is blue. Then, since g, a and j are blue, l must be red. Then k has to be blue. This is finally a contradiction, since f, j, k and h are now all blue, proving the theorem.

We included this proof to indicate the potential for using the triangular lattice to prove such results. We turn to this question in more detail later. Actually we have a stronger result, replacing E^3 by E^2.

Theorem 1′. *If the points of* E^2 *. are 2-colored, (red and blue), then either there is a red pair* 1 *apart, or* 4 *blue points on a line* 1 *apart.*

Proof. Suppose E^2 is 2-colored so that neither a red pair distance 1 apart nor a blue four on a line distance 1 apart occurs. Then there must be a red point p somewhere in E^2. Consider the circle C_1 of radius 1 and center p. The points of the circle must be entirely blue. Now consider the concentric circle C_2 with center p and radius $\sqrt{3}$ (see Figure 2).

In Figure 2, we have the two concentric circles with an equilateral triangle a, b, c inscribed in C_2. The distances between a and d, d and e, and e and b are all 1. d and e are both blue as they are on C_1. Thus not both a and b are blue. This is true similarly for a and c or b and c. Hence at most one of a, b, c is blue. Suppose a and b are red. Let the points f and g be distance 1 both in the clockwise direction from a and b, respectively, and on C_2. Then the points f and g are blue, as are h and i, which are on C_1. This is a contradiction, as the distances fh, hi, ig are all 1.

Theorem 2. *If the points of* E^3 *are 2-colored (red and blue), then either there is a red* $(1, 1, \sqrt{2})$*-triangle, or a blue unit square.*

Proof. If there are no red points, it is trivially true. So let $\boldsymbol{a}$ be red. Suppose there is no red $\boldsymbol{b}$ at distance 1 from $\boldsymbol{a}$. Then the unit sphere centered at $\boldsymbol{a}$ is all blue. A unit square is imbeddable in the unit sphere, and the theorem is true in this case.

Finally, then, we can assume $\boldsymbol{a}$ and $\boldsymbol{b}$ are both red and distance 1 apart. Consider the two circles of radius 1 with centers at $\boldsymbol{a}$ and $\boldsymbol{b}$ respectively, and perpendicular to the line $\boldsymbol{ab}$. If a point $\boldsymbol{c}$ on one of these circles is red, then $\boldsymbol{abc}$ is a red $(1, 1, \sqrt{2})$-triangle. Thus we can assume both circles are blue. But then we clearly can find a blue unit square by taking two points each circle. This completes the proof.

The same result holds if "square" is replaced by "rectangle whose longer side is 1".

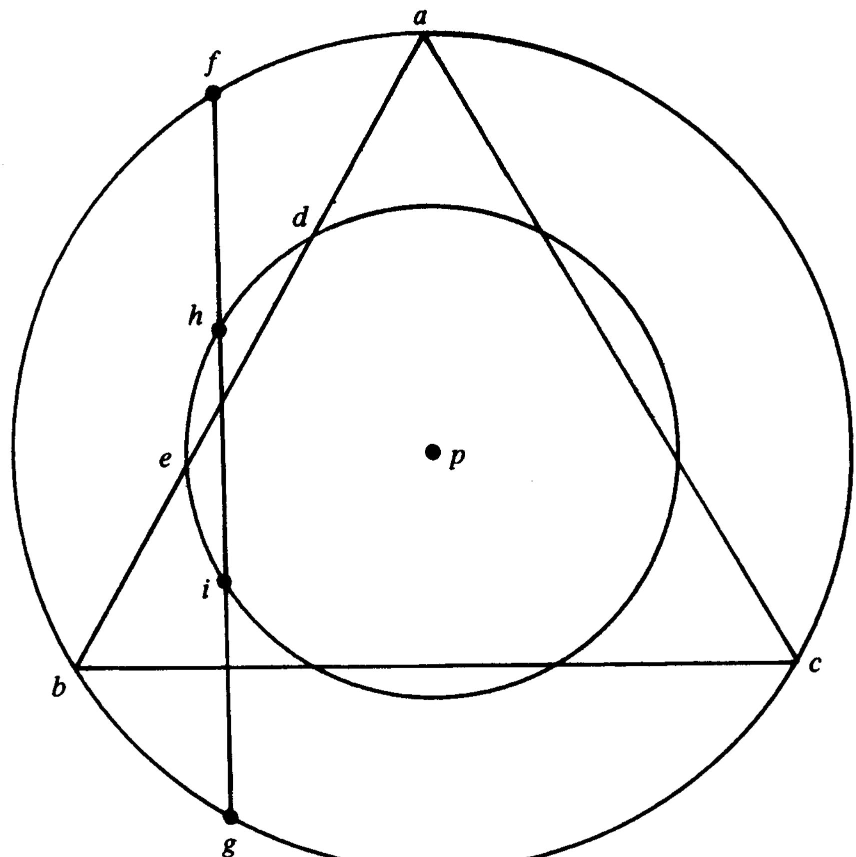

Figure 2

Theorem 3. *Let* T *be any three points in* E^2. *2-color the points of* E^2 *(red and blue). Then either there are two red points distance* d *apart, or a blue translate of* T.

Proof. Let t_1, t_2, t_3 be the points of T, and let K be the seven point configuration in Figure 3. If there is no pair of red points distance d apart, each $t_i + K$ can have only two red points. Hence for some $k \in K$, $T + k$ is blue. This completes the proof.

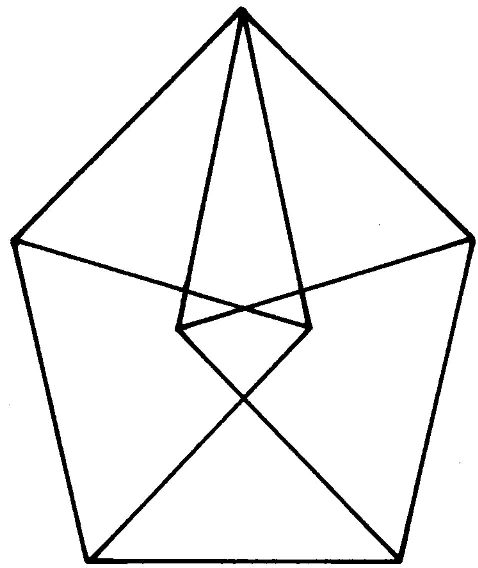

Figure 3
All edges have length d

In Corollary 6 below we generalize Theorem 3 considerably. Whereas Theorem 3 says we must have either a red unit distance or a blue translate of a set K, Corollary 6 says we must have a red brick B' congruent to B, a fixed brick, or a blue translate of K (in E^m for sufficiently large m). In particular, since all Ramsey sets known so far are subsets of vertices of bricks, we know that for all known cases the following statement is true, and we conjecture that it is true in general: Let K be a set of k points, and let B be a Ramsey set. Then there is an n, such that if E^n is 2-colored, either there is a red B' congruent to B, or a blue translate of K.

This may be true even if it turns out that some Ramsey sets are not imbeddable in a brick.

Theorem 1′ and 3 lead to the question: if K is a set in E^2, and the points of E^2 are colored red and blue so that no two red points are distance d apart, then is there necessarily a blue K' congruent to (or a translate of) K? This question is settled negatively by the following example. Let $d = 1$ and let $K =$ the set of 10^{12} points in a regular $10^6 \times 10^6$ lattice of points where the distance between adjacent points is $\frac{3}{10^6}$. Now for each pair of integers m and n, let the square $\{(x, y) \mid 2m \leqslant x \leqslant 2m + 1/2,\ 2n \leqslant y \leqslant 2n + 1/2\}$ be colored red, and let all other points in E^2 be colored blue. Clearly with this coloring there are no red points 1 apart, and no blue K. This same kind of counterexample can be used for the analogous questions in higher dimensions. There is an open question here, which we think should have a negative answer: given k, is there an n, depending only on k, such that if K is a subset of E^k, and E^n is 2-colored so that there is no red pair of points distance 1 apart, then must there be a blue $K' \cong K$? The point here is that n is independent of $|K|$. When it is allowed to depend on $|K|$ also, then Theorem 3 gives an affirmative answer. As a special case we would like to know if there is for each n a set $K_n \subset E^1$ and a 2-coloring of E^n with no red points distance 1 apart and no blue set $K'_n \cong K_n$.

Other open questions arise from Theorems 1 and 2. From Theorem 2 we see that if we 2-color E^3 so that no two red points have distance 1, then there will be a blue unit square. Whether E^2 suffices here is still undecided. From Theorem 1′ we see that any 2-coloring of E^2 yields either a red pair distance 1 apart, or four blue points in a row 1 apart. We don't know whether we can get five points in a row. It is also not known whether we can get five in E^3. However, for E^4 we can, by considering the 4-dimensional generalization of Figure 3 and using a proof analogous to that of Theorem 1.

Some of the asymmetric results can yield statements about the chromatic numbers of certain kinds of graphs. A graph G will be called an *$(n, 1)$-graph* if the vertices of G can be put in a one-to-one correspondence with a set of points G' in E^n such that a pair of vertices in G is connected by an edge in G if and only if the corresponding pair of points in G' are distance 1 apart. For example the graph in Figure 3 is a (2, 1)-graph. Let K be a set of k points in E^n, and let G be an $(n, 1)$-graph with chromatic number $k+1$. Then just as in the proof of Theorem 3, we let G' be a realization of G in E^n, and we consider all the points in $G' + K$. If we 2-color E^n, red and blue, such that no red points are distance 1 apart, then we must have a blue translate of K. Thus if we can color E^n so as to avoid both red points distance 1 apart and blue copies of K, it follows that no $(n, 1)$-graph has chromatic number $k+1$ (or more). In particular, if E^2 can be 2-colored so as to avoid both a red pair of points of unit distance and a blue unit square, then we can conclude that every (2, 1)-graph has chromatic number 4 or less.

3. DENSITY ARGUMENTS AND SPECIAL SETS

The arguments in the previous sections which gave positive Ramsey theorems about various sets were all proved by constructing (even if only inductively) an appropriate finite set, the coloring of which forced one of the monochromatic configurations under consideration (see Proposition 4 of [2]). It is possible in some cases to construct sets so rich in some special configuration that simply by counting we can verify the existence of monochromatic instances of the configuration. The most obvious case of such a set, of course, is the simplex. If we consider the simplex on n points, then any fraction α of them where $\alpha \geqslant \frac{r}{n}$ must contain an r point simplex. Thus we are guaranteed an r-point monochromatic simplex using $[n/r]$ colors. For configurations other than simplicial ones the required sets may be complicated. The fractions obtained in these cases give bounds on the number of colors required in a given dimension to prevent the configuration from occurring monochromatically. In particular, if in E^m we can find a set S with n points such that every subset of size n^ϵ, $\epsilon < 1$, contains the given configuration, K, then $R(K, m, [n^{1-\epsilon}])$ holds. The following construction is a generalization of a construction of Jon Folkman. A. Hajnal and E. Szemerédi showed that it is the best possible for constructions of this type (partitioning coordinates).

Theorem 4. *Let B be a k-dimensional brick, $k \geqslant 1$, with dimen-*

sions $d_1, d_2, \ldots, d_k$. *Then there is a set* S *of* N *points in* E^m *such that every subset of* S *with* $2^k N^{2^k - 2/(2^k - 1)}$ *points contains a brick congruent to* B, *where* $N = n^{2^k - 1}$, $m = n^{2^k}$, n *is any integer greater than* 1.

Proof. Let S be the set of points

$$(x_{11}, \ldots, x_{1n}, x_{21}, \ldots, x_{2n^2}, x_{31}, \ldots$$
$$\ldots, x_{3n^4}, \ldots, x_{k1}, \ldots, x_{kn^{2^{k-1}}})$$

where for each $j = 1, 2, \ldots, k$ all the x_{ji} are 0 except for one, which must be $1/\sqrt{2} \cdot d_j$. There are $N = n^{2^k - 1}$ of these points in $(n^{2^{k-1}} + \ldots \ldots + n^2 + n)$-dimensional space, and thus in $m = n^{2^k}$-dimensional space.

Now consider the sets $A_j = \{j, i \mid i = 1, 2, \ldots, 2^{j-1}\}$, $j = 1, \ldots, k$. Let T be the set of all k-tuples consisting of one element from each of the A_j. Each element $t \in T$ corresponds to a point $s \in S$ by choosing $x_{ji} \neq 0$ in s if and only if $(j, i) \in t$.

Select two elements form each of the A_j and form the subset of T consisting of all the k-tuples in T composed only of the selected elements. Call such a set a 2^k-subset of T. (It has 2^k k-tuples in it.) From the construction of S, each 2^k-subset of T corresponds to a brick in S congruent to B. So what we wish to show is that any $2^k n^{2^k - 2}$ of the $n^{2^k - 1}$ elements of T contain a 2^k-subset.

We can do this by induction on k. For $k = 1$ the result is trivial, since every two elements of A_1 correspond to a pair of points at distance d_1 (or 1-dimensional brick).

We consider $k = 2$ next to illustrate the idea in the general case. In this case T can be thought of as the complete bipartite graph K_{n,n^2}. Suppose T has a subset T' without 4-cycles (quadrilaterals). If we count pairs of edges of T' with common vertex in A_2, we get $\binom{n}{2}$ at most, for otherwise T' would contain a quadrilateral. There are then at most

$2\binom{n}{2}$ edges contained in such pairs. There may be at most n^2 edges in T' not in such a pair. Thus there are at most $2\binom{n}{2} + n^2 = 2n^2 - n$ edges in T'. Any subset of T with at least $2n^2$ edges therefore contains a quadrilateral. This completes the $k = 2$ case, since such quadrilateral correspond to bricks.

We next assume that the statement is true for k and show it for $k + 1$.

Consider $A_1, A_2, \ldots, A_{k+1}$ and T as defined above. Suppose T' is a subset of T not containing a 2^{k+1}-subset. We count pairs of $(k + 1)$-tuples in T' with a common point in A_1. If $a \in A_1$, then the set of $(k + 1)$-tuples in T' containing a deteremines a set of k-tuples with one element from each of the A_j, $j \geqslant 2$. Let this set be $T'(a)$. If a and b are in A_1 and $|T'(a) \cap T'(b)| \geqslant 2^k (n^2)^{2^k - 2}$, then by induction (with n^2 replacing n), $T'(a)$ and $T'(b)$ have a common 2^k-subset, U. But this in turn gives us a 2^{k+1}-subset of T' consisting of all those $(k + 1)$-tuples $\{a\} \cup U_k$ and $\{b\} \cup U_k$ for all $U_k \in U$. This is impossible by choice of T'. So $|T'(a) \cap T'(b)| < 2^k n^{2^{k+1} - 4}$.

If we now count all pairs of $(k + 1)$-tuples of T' differing only in A_1, we get at most $\binom{n}{2} 2^k n^{2^{k+1} - 4}$. (There are $\binom{n}{2}$ ways to pick the element in A_1 for such a pair, and at most $2^k n^{2^{k+1} - 4}$ 2^k-sets in the $A_2, A_3, \ldots, A_{k+1}$ in which they can overlap.) This then accounts for $n(n - 1) 2^k n^{2^{k+1} - 4}$ of the $(k + 1)$-tuples in T'. There are at most $(n^2)^{2^k - 1} = n^{2^{k+1} - 2}$ members of T' not in any such pair. Hence T' can have at most $n(n - 1) 2^k n^{2^{k+1} - 4} + n^{2^{k+1} - 2}$, and the induction is complete.

We note then that Corollary 21 of [2], which says that any brick is Ramsey, is also a corollary of this theorem, since as we observed before, Theorem 4 implies $R(B, n^{2^k}, [n/2^k])$. Theorem 4 can also be used to generalize Theorem 3 as follows.

Theorem 5. *Let S be a set of N points in E^m so that every subset of $[N/r]$ points of S contains a set K' congruent to a given set K. Let $l < r$, and let L be a set in E^m with l points. Then if E^m is 2-colored (red and blue), either there is a red K' congruent to K or a blue translate of L.*

Proof. Consider each set $a + S$ for $a \in L$. If E^m is 2-colored to avoid a red K, then fewer than $[N/r]$ of the points of $a + S$ are red. Thus, since $l[N/r] < N$, there must be some point $s \in S$ such that $a + s$ is blue for all $a \in L$. This gives a blue L' which is a translate of L, completing the proof.

Let $n > 1$ and $k > 0$ be integers, and let $m = n^{2^k}$, $N = n^{2^k - 1}$, and l and integer less then $[n/2^k]$. Let L be a set in E^m with l points, and let B be a k-dimensional brick. Then from Theorems 5 and 4 we get

Corollary 6. *If E^m is 2-colored (red and blue), then either there is a red B' congruent to B or there is a blüe translate of L.*

Since the only Ramsey sets we know so far are embeddable in bricks, it is suggested by the arguments above that perhaps the only Ramsey sets are those which satisfy a similar density property. That is, if K is Ramsey, must there be an n element set S_n for large n such that any subset of S_n with $f(n)$ points contains a K' congruent to K, and $f(n) = o(n)$?

In the case of sets which are not Ramsey we obtain the inverse statement. That is, for example, if K consists of three points equally spaced on a line, then we have for every set S_n of n points some subset of at least $n/4$ points which contains no K' congruent to K, since $R(K, n, 4)$ is false (see [2] Sec. 3).

Another conjecture is that if K is a configuration associated with a function $f(n)$ as described above, and if in fact $f(n) = o(\sqrt{n})$, then K must be a simplex. With the square we got $f(n) = O(n^{2/3})$, and with the 1, 1, $\sqrt{2}$ right triangle we got $O(\sqrt{n})$, as seen below.

Theorem 7. *There is a set* S *in* E^n *with* $\frac{n^2 - n}{2}$ *points such that any subset of* S *with* n *points contains a* $(1, 1, \sqrt{2})$*-triangle.*

Proof. The set S we choose is the set of all points $(x_1, \ldots, x_n)$ where all the x_i are 0 except for two, which are $1/\sqrt{2}$. There are $\binom{n}{2}$ of these. If we consider the complete graph K_n on n vertices, where the edge between vertices i and j corresponds to the point $(x_1, \ldots, x_n)$ with $x_i = x_j = 1/\sqrt{2}$, then clearly non-cyclic paths of length three in K_n correspond exactly to $(1, 1, \sqrt{2})$-triangles. Since the only subset of edges of K_n without paths of length three are disjoint triangles and stars, any set of $n + 1$ edges in K_n must contain a path of length three. Thus any $n + 1$ points of S contain a $(1, 1, \sqrt{2})$-triangle, completing the proof.

The construction used in Theorem 7 gives a set S of $\binom{n}{2}$ points such that there are $12\binom{n}{4}$ $(1, 1, \sqrt{2})$-triangles among them. This is more than $\left(2 - \frac{8}{n}\right)|S|^2$. What is the largest number of $(1, 1, \sqrt{2})$-triangles we can have in a point set of size N in a Euclidean space? We ask the same question for the unit square. In that case, the construction in Theorem 4 gives about $N^2/4$. We can in fact do better in both of these cases.

Theorem 8. *There is a set* S *of* $N = (3/2)n$ *points in* E^{n+2} *such that, for* n *sufficiently large, at least* $(1/15)N^3$ *of the triples form* $(1, 1, \sqrt{2})$*-triangles.*

Proof. Let $A = \{x_1, \ldots, x_n, y, z)|$ all entries are 0 except for one of the x_i, which is $1\}$, and let $B = \{1/n, \ldots, 1/n, y, z)|\ z^2 = 1/n\}$. The circle B is distance 1 from all points in A. Let B_α be any set of αn points of B, and let $S = A \cup B_\alpha$. S has $(1 + \alpha)n$ points, and any two from A with one from B from a $(1, 1, \sqrt{2})$-triangle. Thus there are $\binom{n}{2}\alpha n$ of these triangles. The maximum value of $\binom{n}{2}\alpha n((1 + \alpha)n)^{-3}$ occurs at $\alpha = 1/2$. Thus if we let $\alpha = 1/2$, $N = (3/2)n$ we get $(2/27)(1 - 1/n)N^3$ triangles. For n large enough, this is more than $(1/15)N^3$ and we are done.

We observe two things about this construction. First, it works equally well for any isosceles triangle. Second, although the construction yields more triangles than that used in Theorem 7, it requires more than $(1/3)N$ of the points to guarantee a triangle, as opposed to $O(\sqrt{n})$ in Theorem 7. If we let K be a configuration on n points maximizing the number of $(1, 1, \sqrt{2})$-triangles, then $\lim_{n\to\infty} \left(\frac{t(K)}{n^3}\right) < 1$, where $t(K)$ is the number of $(1, 1, \sqrt{2})$-triangles in K. The value of this limit is unknown. If S is a set of n points maximizing the number of unit squares, the next theorem gives $q(S) \geqslant n^2$, where $q(S)$ is the number of unit squares in S. We do not know whether $q(S) = o(n^3)$.

Pósa asked whether in Hilbert space every set of c points has a aubset of c points without any right triangles. In E^n the answer is affirmative.

Theorem 9. *There is a set* S *of* $N = \binom{m}{2\left[\frac{\sqrt{m}}{2}\right]}$ *points in* E^m *such that for* m *sufficiently large.* S *contains at least* N^2 *unit squares.*

Proof. For $k \leqslant m/4$, let S_k denote the set $\{(x_1, \ldots, x_m) \mid$ exactly $2k$ of the x_i are $1/\sqrt{2k}$ and $m - 2k$ are $0\}$. We can form unit squares in S_k by choosing four disjoint k-sets of coordinates A, B, C, D, and choosing the four points whose non-zero entries occur precisely in $A \cup B$, $B \cup C$, $C \cup D$ and $D \cup A$, respectively. There are $Q =$ $= \left(\frac{1}{8}\right)\binom{m}{k}\binom{m-k}{k}\binom{m-2k}{k}\binom{m-3k}{k} = \left(\frac{1}{8}\right)\binom{m}{2k}\binom{m-2k}{2k}\binom{2k}{k}^2$ of these. Now

$$Q\Big/\binom{m}{2k}^2 = \left(\frac{1}{8}\right)\binom{m-2k}{2k}\binom{m}{2k}^{-1}\binom{2k}{k}^2 > \left(\frac{1}{8}\right)\left(\frac{m-2k}{m}\right)^{2k}\binom{2k}{k}^2 .$$

Let $k = \left[\frac{\sqrt{m}}{2}\right]$. Then since $\left(\frac{m-2k}{m}\right)^{2k} \cong \left(1 - \frac{1}{\sqrt{m}}\right)^{\sqrt{m}}$ and this goes to $1/e$ as $m \to \infty$, we can pick m large enough so that $\left(\frac{m-2k}{m}\right)^{2k} > 1/3$, or $Q\Big/\binom{m}{2k}^2 > \left(\frac{1}{24}\right)\binom{2k}{k}^2$. Clearly for m large enough $\binom{2k}{k}^2 > 24$. Thus we have at least $\binom{m}{2k}^2 = N^2$ unit squares, and the theorem is proved using $S = S_{[\sqrt{m}/2]}$.

The arguments above all involved the construction of some special set which was "dense" in a particular configuration. In fact, these constructions were all based more or less directly on the simplices. Pósa proved a result for infinite sets of this type. Namely, there exists a set of power c in Hilbert space so that any subset of power c contains a set similar to the infinite unit simplex. He used C.H. We don't know any other sets for which this is true. We can restrict our attention to prescribed types of sets and ask about the density of various configurations contained in them. In particular, in view of some of the arguments used before (cf. Theorem 1) we might consider regular lattices, such as the triangular lattice or the square or rectangular lattices.

Suppose K is the configuration of three points on a line distance one apart. Then in the m-dimensional integer lattice of side n,

$$L_{m,n} = \{(x_1, \ldots, x_m) \mid 1 \leqslant x_i \leqslant n,\ 1 \leqslant i \leqslant n\},$$

we can find a set S with $(2/3)n^m$ points (asymptotically with n) containing no K' congruent to K. Just let

$$S = \Big\{(x_1, \ldots, x_m) \mid 1 \leqslant x_i \leqslant n,\ 1 \leqslant i \leqslant m,\ \sum_{i=1}^{m} x_i \not\equiv 0 \pmod 3\Big\}.$$

This is clearly the largest number possible, for if we divide $L_{m,n}$ into disjoint, adjacent triples in, say, the $(1, 0, \ldots, 0)$ direction, we must exclude at least one from each triple.

In the simplicial lattice we don't yet have such a complete answer. Let $0, \nu_1, \ldots, \nu_m$ be the points of a simplex in E^m, and let $S_{m,n} = \{i_1\nu_1 + \ldots + i_m\nu_m \mid 1 \leqslant i_j \leqslant n,\ 1 \leqslant j \leqslant m\}$. If $m = 2$, this is just the triangular lattice in the plane, and for this case we obtain the set

$$S = \{i\nu_1 + j\nu_2 \mid i - j \not\equiv 0 \pmod 3,\ 1 \leqslant i, j \leqslant n\}.$$

S has no K' congruent to K, and $|S| = 2/3n^2$ (asymptotically in n). It is undecided whether there is such a set with $2/3n^m$ points in the lattice $S_{m,n}$. Since K is not a Ramsey configuration, and in fact $R(K, n, 4)$

is false, we know that we can find a set S with more than $(1/4)n^m$ points containing no K' congruent to K.

We might get a larger number of some configurations K in a lattice by reducing the side of the lattice, or, equivalently, increasing the scale of K. For instance, in the $n \times n$ integer lattice $L_{2,n}$ there are $2n(n-1)$ pairs distance 1 apart. But there are $2n(n-5) + 2(n-3)(n-4)$ pairs of points distance 5 apart, since some diagonals have length 5. For some configurations K, however, changing scale doesn't help. An interesting case is the equilateral triangle. Clearly there is no equilateral triangle of side 1 in $L_{2,n}$. As a special case of the next theorem we see that $L_{2,n}$ contains no equilateral triangle of side d either.

Theorem 10. *Let K be any odd polygon (we permit edges to cross) with all sides of length d. Then the integer lattice $L_{2,n}$ contains no K' congruent to K.*

Proof. Let d be realizable as the diagonal of the integer rectangles with horizontal side length a_i and verticle side length b_i, $1 \leqslant i \leqslant k$. Let P be a closed polygon of side length d in $L_{2,n}$ with m sides. If we start at one vertex of P and move around the polygon along successive edges, each step contributes to the horizontal and vertical components of the position. Let the successive contributions be (x_i, y_i), $1 \leqslant i \leqslant m$. Since P is a closed polygon, the total contributions must both be 0. That is $\sum_{i=1}^{m} x_i = \sum_{i=1}^{m} y_i = 0$. The pairs (x, y) must be chosen from the pairs (a_i, b_i).

Now let q be the largest integer so that 2^q divides all the a_i and b_i, and let $(a_i, b_i) = 2^q(a_i', b_i')$. By considering $d^2/2^{2q} \pmod 4$ we see that if one pair (a_i', b_i') has one odd and one even number, so do all the other pairs (a_j', b_j'). In this case, if m is odd there are either an odd number of $x_i \equiv 2^q \pmod{2^{q+1}}$, or an odd number of $y_i \equiv 2^q \pmod{2^{q+1}}$, an impossibility since the x's and y's sum to 0. If there are no even-odd pairs (a_i', b_i'), then all a_i' and b_i' are odd, and we get the same contradiction as above if m is odd. But by the choice of q, these are the only possibilities. Thus m is even, and the theorem is proved.

4. INFINITE CONFIGURATIONS

So far our concern has been only with finite configurations which are or are not necessarily monochromatic in certain colorings. We now turn our attention to the question of infinite sets. As we might expect, many of the direct analogues to finite theorems are false in the infinite case. However, some are true. We recall as an illustration that the infinite version of Ramsey's theorem is true, while the infinite version of van der Waerden's theorem is false. These observations imply directly the following two theorems.

Theorem 11. *Let* $E^{\aleph_0}$ *denote* $\aleph_0$*-dimensional Euclidean space (=all* $\aleph_0$*-tuples with only finitely many nonzero terms, and with the usual metric). Let* K *be the* $\aleph_0$*-simplex,* $\{y \mid y$ *has exactly one coordinate equal to* 1 *and all others* $0\}$*. If the points of* $E^{\aleph_0}$ *are r-colored in any way (in fact the points of* K *are all that's necessary to consider here), then there must be some monochromatic* K' *congruent to* K*.*

We use the term "congruent" above, and will below, to mean that K' is the image of K under an isometric mapping f of $E^{\aleph_0}$ into $E^{\aleph_0}$ which is not necessarily onto. If we required f to be onto as well, then K' would have to have the same codimension as K, and Theorem 11 would not be true. This same convention is used for the notion of similarity also when appropriate.

Theorem 12. *If* $\mathcal{H}$ *is real Hilbert space, then* $\mathcal{H}$ *can be 2-colored so that there is no monochromatic set* K' *congruent to any arithmetic progression (or, alternatively, similar to* $K = \{(i, 0, 0, \ldots) \mid i = 0, 1, 2, \ldots\}$*).*

Proof. We decompose $\mathcal{H}$ into concentric shells around the origin with thicknesses increasing and unbounded. Then we color the shells alternately. This generalizes the usual 1-dimensional coloring.

Besides the kinds of variations we considered for the finite case, such as the asymmetric problems and problems with different groups, there are in the infinite case some kinds of questions involving analytical or topological properties not relevant in the finite case.

Theorem 13. *Let* G *be the group of homeomorphisms of* E^2 *onto itself. If* K *is any discrete set in* E^2, *then* $R_G(K, 2, 2)$ *is true. If* K *is any set which is everywhere dense, then* $R_G(K, 2, \aleph_0)$ *is false.*

Theorem 14. *There is a set* S *of* c *(continuum) real numbers such that* E^1 *can be 2-colored with no monochromatic pair of points distance* d *apart for any* $d \in S$. *No such* S *of positive measure exists.*

Proof. For the first part, any set S of c numbers which are rationally independent will work. For let B be a Hamel basis containing S. Let G be the additive subgroup $G = \langle B, + \rangle$ of the reals generated by B. We color the elements of G according to the parity of the sum of the "coefficients" of the elements of B. That is, if $g = \sum_{i=1}^{k} n_i b_i$, $b_i \in B$, $n_i \in Z$, then b is colored according to the parity of $\sum_{i=1}^{k} n_i$. This colors G. Now for each coset $r + G$ of G, choose a representative r, and color each element $r + g$ the same as g. E^1 is now all colored. The only points x, y which can be a distance $b \in B$ apart are points x, y in the same coset of G. But if $x - y = b \in B$, then the parity of x and y differ. Thus no distance in $S \subseteq B$ occurs monochromatically.

Suppose S has positive measure. Let E^1 be 2-colored with no monochromatic distance $d \in S$. Suppose the color of 0 is red. Then the set $S \subseteq E^1$ is blue. Then $S + S = \{x \mid x = s_1 + s_2, s_1, s_2 \in S\}$ must be all red. By a well-known theorem, since S has positive measure, $S + S$ must contain an interval $\{a, b\}$. Thus we have shown that if 0 is red, $[a, b]$ is red. Similarly then, using each point in $[a, b]$ as a new origin, we get $[2a, 2b]$ all red. In general then, arbitrarily large intervals are all red a contradiction completing the proof.

Theorem 15. *We can 2-color* E^1 *so that no pair of red points are a rational distance apart, and no set of blue points in congruent to the set of rational numbers.*

Proof. Again we appeal to a Hamel basis B, where we take $1 \in B$. Color red any point in E^1 whose Hamel basis expansion has a 0 coefficient for 1. Color all other points blue. Clearly no two red points differ by a rational distance. On the other hand, suppose some set $x + Q$ was all blue. Then x cannot be rational (as 0 is red), and thus $x = x_0 + \sum_{i=1}^{k} x_i b_i$ where the $x_i \in Q$ and $b_i \in B$. Then we have $\sum_{i=1}^{k} x_i b_i \in x + Q$. This is a contradiction since $x + Q$ is all blue, while $\sum_{i=1}^{k} x_i b_i$ must be red by the choice of the coloring. This completes the proof.

It is an open question whether this can be extended to E^2. We might extend it in any of several ways, actually. We may try to color E^2 to have no red pair of points a rational distance apart and no blue set congruent to Q in E^2. We may replace Q by $Q \times Q$, or perhaps by the sets A or C of algebraic or constructable points, respectively.

Theorem 16. *Given any set K with $|K| > 1$ in any vector space L, we can color the points of L in two colors so that there is no monochromatic translate of K.*

Proof. The proof of this is similar to that of Theorem 14 above. Let G be the additive subgroup of L generated by K. G is countable. Each coset of G in L is a translate of G in L, and vice-versa. Each translate $l + K$ of K in L is in the coset $l + G$ of G. Just as in the proof of Theorem 14, we need only color the elements of G to avoid monochromatic translates of K, and then color each element $l + g \in l + G$ the same color as g.

It is sufficient to let $K = \{0, k\}$. Then we color nk according to the parity of n. This coloring clearly avoids any monochromatic translate of K, completing the proof.

We see that $R_G(K, n, r)$ is therefore uninteresting for K infinite and G the group of translations. Theorem 16 can be made more general by replacing L by any group. We can also make a similar asymmetric argument excluding k fixed distances in red and a translate of K in blue.

Theorem 12 is a negative result about sets similar to $K = \{0, 1, 2, 3, \ldots\}$, i.e. arithmetic progressions. A stronger result with essentially the same proof follows.

Theorem 17. *Let K be an unbounded set in Hilbert space $\mathcal{H}$. Then $\mathcal{H}$ can be 2-colored (in concentric shells) so that no K' similar to K is monochromatic.*

We also get negative results for some asymmetric problems in the finite dimensional case.

Theorem 18. *E^2 can be 2-colored so that no two blue points are distance 1 apart and no infinite arithmetic progression is entirely red.*

Proof. Let A_i be the open annulus around the origin with inner and outer radii 10^{2i} and 10^{2i+1}, $i = 1, 2, \ldots$. Let C_1 be the set of all discs of radius $1/4$ contained in any of the A_{2^i} which have centers on points of the form $(10m, 10n)$, m, n integers. By the Kronecker approximation theorem, every arithmetic sequence contained in a line of irrational slope meets some disc in C_1.

Let (a_i, b_i, c_i), $i \geq 2$ be all triples of rational numbers. For each $i \geq 2$ let p_i be the i-th prime number. Let C_i denote the set of all discs with radius $1/4$ contained in any $A_{p_i^j}$, $j \geq 1$, with centers on points of the form $(a_i, b_i) + 10n(1, c_i)$, n and integer. Then any infinite arithmetic progression with slope c_i and initial point in a neighbourhood of (a_i, b_i) (say a disc of radius $1/10$) must meet some disc in C_i. Thus $C = \bigcup_{i=1}^{\infty} C_i$ is a set of discs such that every arithmetic sequence meets one of them (they actually meet an infinite number).

All the discs in C are of diameter $1/4$, and all are at least distance 10 apart. Thus if we color all points in the discs of C blue and everything else red, the proof is complete.

Theorem 18 can be generalized by induction to E^n as follows. Taking concentric shells A_i as above, and using a lattice of n-balls instead of discs, we obtain a set C_1 of n-balls which meet every arithmetic progression with "totally irrational" slope. That is, if $u + iv$, $i = 1, 2, \ldots$ is the progression, the coordinates of v are rationally independent. For each of the denumerably many possible rational relations between the coordinates of v, we get an m-dimensional problem, $m < n$. For this we have a set of n-balls which meet all such sequences. Thus we can choose a countable set of n-balls $\bigcup_{i=1}^{\infty} C_i$ meeting all arithmetic progressions. We let the C_i be in alternate A_i as above.

What we have here, then, is an asymmetric result allowing us to color E^n to avoid a blue unit distance and any red K' similar to $K = \{1, 2, 3, \ldots\}$. The questions of replacing K by an arbitrary infinite set, and E^n by Hilbert space are open. It appears that for some K in E^n a set of small blue discs far apart may not be sufficient to provide a suitable coloring, as it was in the case of arithmetic progressions. Certainly for some sequences K which are "close" to arithmetic sequences the result of Theorem 18 can be extended. What seems particularly interesing is the case where K is a convergent sequence, like $\left\{\frac{1}{n}\right\}$. For some sets, such as $K = \{x \mid |x| = 1\}$, the result of Theorem 18 clearly doesn't hold, for such a red set K must surround every blue point. Perhaps for K unbounded the generalization of Theorem 18 holds. Replacing similarities by congruences or other groups also leads to new questions. Finally we comment: If E^n (or $\mathscr{H}$) is 2-colored so that no two blue points are distance one apart, then if a set K occurs in blue, a set $K' \cong K$ occurs in red. K' can be taken as any unit translate of K.

Returning to the symmetric case, we obtain a result stronger than Theorem 17.

Theorem 19. *Let K be any infinte set in E^n. Then E^n can be $\aleph_0$-colored so that every K' similar to K contains points of every color.*

Proof. Let k be the smallest integer such that K has a denumerable subset L_1 contained in some k-dimensional Euclidean subspace, E^k. Let L_2 be a denumerable subset of L_1 so that no $k + 1$ points of L_2

are in a $(k-1)$-dimensional Euclidean subspace. (Such a set exists by the choice of k.) Finally, we choose inductively a set $L = \{y_1, y_2, \dots\} \subseteq L_2$ such that no two subsets of $k+2$ points are similar. This can always be done one step at a time as follows. Suppose we have chosen $S_n = \{v_1, \dots \dots, v_n\}$ satisfying these condictions. If X_{k+1} and Y_{k+2} are a $(k+1)$-set and a $(k+2)$-set in L_2, respectively, then there are only a finite number of points v which can make $X_{k+1} \cup \{v\}$ similar to Y_{k+2}. And if X_{k+1} and Y_{k+1} are both $(k+1)$-sets, there are still only a finite number of points in E^k for which $X_{k+1} \cup \{v\}$ and $Y_{k+1} \cup \{v\}$ can be similar. Since there are only a finite number of possible X_{k+1} and Y_{k+2} at each step, and since L_2 is infinite, we can always choose v_{n+1}. Thus no two $(k+2)$-subsets of L are similar.

Then any two sets L' and L'' which are similar to L must contain at most $k+1$ points in common, since L is determined by any $k+1$ of its points. By a theorem of Erdős and Hajnal on families of sets with bounded intersections [3], the points of E^n can be $\aleph_0$-colored so that every L' similar to L contains points of every color. This completes the proof.

We see by Theorem 11 that for infinite dimensional spaces Theorem 19 isn't true. However, if K is a subset of $E^n \subset E^{\aleph_0}$ for some n, then the proof of Theorem 19 would apply. The Erdős – Hajnal result will not apply in general because the set L which was fixed by any $k+1$ of its points may not exist. We remark finally that similarities could be replaced by affine motions in Theorem 19, because we could still find some m and set L with every m points of L fixing L.

Let $f(i)$ be a positive real-valued function on the positive integers. In $E^{\aleph_0}$ let x_i denote the point with the i-th coordinate equal to $f(i)$ and all other coordinates equal to 0. Let x_0 be the origin, and let $K_f = \{x_0, x_1, x_2, \dots\}$. We recall that a set K is a space E is called Ramsey if for every $r > 0$ and every r-coloring of E there is a monochromatic K' congruent to K. For the K_f above we have the following result, which generalizes Theorem 11.

Theorem 20. *If f is a bounded function then K_f is Ramsey in* $E^{\aleph_0}$.

Proof. Let $E^{\aleph_0}$ be r-colored for some $r > 0$. Suppose that no $K' \cong K_f$ ($\cong$ means "congruent to") is monochromatic. If p_0 is any point in $E^{\aleph_0}$, we try to choose sequentially points $p_1, p_2, \ldots$ such that all the p_i and p_0 are the same color, $(p_0 - p_i)$ and $(p_0 - p_j)$ are orthogonal for $i \neq j$, and $|p_0 - p_i| = f(i)$. If we could continue choosing the p_i indefinitely we would have a monochromatic $K' \cong K_f$. Thus for some $p_1, \ldots, p_m$ we can't continue. Let S be the sphere with center p_0 consisting of all points x with $(p_0 - x)$ orthogonal to $(p_0 - p_i)$ for all $i = 1, 2, \ldots, m$, and with $|p_0 - x| = f(m+1)$. Then all points on S are colored differently from p_0.

Thus we can associate with each point p in $E^{\aleph_0}$ a sphere $S(p)$ with radius $R(p) \leqslant \sup_i f(i)$, of finite codimension such that the color of any point in $S(p)$ is different from the color of p. What we wish to do now is to find points $p_1, p_2, \ldots, p_{r+1}$ with $p_{i+1} \in S(p_1) \cap \ldots \cap S(p_i)$ for $i = 1, 2, \ldots, r$. This would be a contradiction since it implies that the colors of p_i and p_j are distinct for $i < j$, while there are only r colors.

We choose the p_i sequentially. Let $\epsilon = 2^{-2^r}$. Let $M_1 = \sup_{E^{\aleph_0}} R(p)$, and let p_1 be some point with $R(p_1)^2 > M_1^2(1 - \epsilon)$. Set $S_1 = S(p_1)$, $r_1 = R(p_1)$, and $s_1 = r_1$. Similarly for $i = 2, 3, \ldots, r+1$ we let $M_i = \sup_{p \in S_1 \cap \ldots \cap S_{i-1}} R(p)$, (we will prove below that $S_1 \cap \ldots \cap S_{i-1} \neq \phi$) and let p_i be some point in $S_1 \cap \ldots \cap S_{i-1}$ with $R(p_i)^2 > M_i^2(1 - \epsilon)$. We let $r_i = R(p_i)$, $S_i = S(p_i)$ and $s_i = \sqrt{r_i^2 - r_i^4/(4s_{i-1}^2)}$, the radius of $S_1 \cap \ldots \cap S_i$. Then the proof is complete if we can show that this construction yields $S_1 \cap \ldots \cap S_r \neq \phi$. But to show this it is sufficient to show that $(s_i/(r_{i+1}))^2 > 1/2(1 - 2^i\epsilon)$, $1 \leqslant i \leqslant r$. We do this by induction. For $i = 1$ it is true by the choice of r_1. In general $(r_i/(r_{i+1}))^2 \geqslant r_i^2/(M_i^2) > (1 - \epsilon)$, by the choice of r_i. Then $(s_{i-1}/(r_i))^2 > 1/2(1 - 2^{i-1}\epsilon)$, $i \geqslant 2$, implies that

$$\left(\frac{s_i}{r_{i+1}}\right)^2 = \left(\frac{r_i}{r_{i+1}}\right)\left(\frac{s_i}{r_{i+1}}\right)^2 = \left(\frac{r_i}{r_{i+1}}\right)^2\left(1 - \frac{1}{4\left(\frac{s_{i-1}}{r_i}\right)^2}\right) >$$

$$> (1-\epsilon)\left(1 - \frac{1}{4 \cdot \frac{1}{2}(1 - 2^{i-1}\epsilon)}\right) =$$

$$= (1-\epsilon)\left(\frac{1}{2} - \frac{1}{2}\frac{2^{i-1}\epsilon}{(1 - 2^{i-1}\epsilon)}\right) =$$

$$= \frac{1}{2}(1-\epsilon) - \frac{1}{2}(2^{i-1}\epsilon)\frac{1-\epsilon}{1 - 2^{i-1}\epsilon} > \frac{1}{2}(1 - 2^i\epsilon) .$$

This completes the proof.

Theorem 20 suggests several questions about Ramsey sets in infinite dimensional spaces. For the finite dimensional case [2] it was true that only sets imbeddable in a sphere could be Ramsey. If we let $f(i) \equiv 1$, then by Theorem 20 K_f is Ramsey in Hilbert space (or in $E^{\aleph_0}$). But K_f is not spherical. Thus not all Ramsey sets are spherical. However, as in the finite case, any set K known to us so far to be Ramsey is a subset of the vertices of a "brick". That is, if f is a real-valued non-negative function on the positive integers, then the set $B(f)$ of all points with the i-th coordinate equal to $f(i)$ for a finite number of i, and equal to 0 for all other i is the brick determined by f. Anything isometric to $B(f)$ is also a brick. The sets K_f in Theorem 20 are subsets of the bricks $B(f)$. In the finite dimensional case we know that all subsets of vertices of bricks are Ramsey. In the infinite case it is an open question whether this is true if f is bounded. (The unbounded case is false by Theorem 17.) In particular, one might ask whether the set K of all points with exactly 2 coordinates equal to 1 and all the other coordinates equal to 0 is a Ramsey set. We construct some other Ramsey sets below by translating Ramsey sets. If the original sets are in a brick, then so are the new ones.

Theorem 21. *Let K be a Ramsey set in a space E. Let L be a finite Ramsey (resp. r-Ramsey (i.e. for r-colors)) set in E^n. Then $K \oplus L$ is Ramsey (resp. r-Ramsey) in $E \oplus E^n$.*

Proof. The proof is the same as in the infinite case. Namely, let $E \oplus E^n$ be colored by r-colors. Let M' be a finite set in E^n such that any r-coloring of M' gives a monochromatic $L' \subseteq M'$ congruent to L. Then if $|M'| = m$ we can r^m-color E by coloring each point $e \in E$ according to the colors of $e \oplus M'$. By the Ramsey property, there is a $K' \subset E$ such that K' is monochromatic in this coloring. But this says that for some $L' \subseteq M'$ congruent to L we have $K' \oplus L'$ all one color. This completes the proof.

We used the crucial fact that for any finite set L which is r-Ramsey, there is some finite set M' such that any r-coloring of M' produces a monochromatic $L' \subseteq M'$ congruent to L. We do not know whether similar things are true for the infinite case. In particular, if K is a countable r-Ramsey subset of $E^{\aleph_0}$ or Hilbert space $\mathscr{H}$, is there a countable set M' so that any r-coloring of M' produces a monochromatic K' isometric to K? All the sets obtained so far have this property. For instance, consider a set K_f from Theorem 20. Let F be the field obtained from the rationals by adjoining the numbers $f(i)$, $1 \leqslant i < \infty$, and let $\bar{F}$ be its algebraic closure. $|\bar{F}| = \aleph_0$. The argument in the proof of Theorem 20 can be carried out in $\aleph_0$-dimensional space over $\bar{F}$.

We also don't know if there is any set $K \subset E^{\aleph_0} \subset \mathscr{H}$ such that K is Ramsey in $\mathscr{H}$ but not in $E^{\aleph_0}$. We can say that K must have less than c points, for, by a transfinite induction argument, we can color $\mathscr{H}$ with c colors so that every copy of K has all c colors if $|K| = c$.

Letting $f(i) = 1/i$, and L the unit simplex in E^n, we get by Theorem 21 that $K_f \oplus L$ is Ramsey. The set of points $0 \oplus L$ are all limit points of $K_f \oplus L$. We have no examples yet of sets K in Hilbert space with an infinite set of limit points, either in K or not in K, such that K is 2-Ramsey. It seems possible that if K has more than $\aleph_0$ limit points, then K cannot be Ramsey.

For instance, any set K which is dense in some line segment is not 2-Ramsey, as we can see by using a coloring by alternate spherical shells of successively smaller thicknesses.

For instance, any set K which is dense in some line segment is not 2-Ramsey, as we can see by using a coloring by alternate spherical shells of successively smaller thicknesses.

A related question concerns ϵ-chains. An ϵ-chain from x to y in some space E is a sequence $x = x_0, x_1, \ldots, x_n = y$ such that $|x_i - x_{i+1}| \leqslant \epsilon$ for all i. Then can we r-color E so that for every pair of points x, y with $|x - y| \geqslant 1$, there is an $\epsilon = \epsilon(x, y)$ so that no ϵ-chain from x to y is monochromatic?

In E^2 this is trivially possible, say, for $r = 9$. Just tile the plane with 2×2 squares, each colored the same way, as 9 differently colored sub-squares $2/3 \times 2/3$. A similar argument will work in E^n with more colors. On the other hand, with $r = 2$ it is impossible for E^2, which we see below. The open question is whether for some finite and some r-coloring of $E^{\aleph_0}$ or $\mathscr{H}$, no points x and y with $|x - y| \geqslant 1$ are connected by monochromatic ϵ-chains for all $\epsilon > 0$.

Theorem 22. *For every 2-coloring of* E^2, *there are points* x, y *with* $|x - y| \geqslant 1$ *such that for any* $\epsilon > 0$, *there is a monochromatic* ϵ-*chain from* x *to* y.

Proof. Let $E^2 = R \cup B$, where R is the set of red points, and B is the set of blue points. Let $\bar{R}$ and $\bar{B}$ be their closures. The components of $\bar{R}$ or of $\bar{B}$ have diameter less than 1. Let S be a 2-sphere in E^3 of radius 1 and tangent to E^2 at 0. Then the stereographic projection of E^2 into S yields sets $\bar{R}_S$ and $\bar{B}_S$, the images of $\bar{R}$ and $\bar{S}$. Letting $\bar{R}'$ and $\bar{B}'$ be closures in S of $\bar{R}_S$ and $\bar{B}_S$, respectively, we have $\bar{R}' = \bar{R}_S \cup \{\infty\}$ and $\bar{S}' = \bar{B}_S \cup \{\infty\}$ where ∞ is the point on S diametrically opposite 0. Furthermore $\{\infty\}$ is a component both of $\bar{R}'$ and of $\bar{B}'$, and all components of $\bar{R}'$ and $\bar{B}'$ have diameter less than 1, $\bar{R}' \cup \bar{B}' = S$.

This leads to a contradiction through an argument in dimension theory (we are indebted to Robert Edwards for showing us this). For there will be a finite number of open, disjoint sets with diameter $\leqslant 1 + \epsilon$ covering S by small open sets with no three having a common point. Considering the "nerve" of this covering leads to the contradiction.

5. EDGE COLORINGS

Suppose that instead of coloring points of E^n we color pairs of points, or "edges". Now we specify a set K of edges in E^n and ask if there is a K' congruent (or similar, etc.) to K which is monochromatic. (By congruent, similar, etc., we mean that the sets of endpoints of the edges are respectively congruent, similar, etc.) Even in the case of similarity, the only non-trivial questions occur when all edges in K have the same length. For suppose some edges of K have length 1 and $\alpha > 1$. Then every positive real number can be represented uniquely in the form $c\alpha^m$ for m an integer, and c a number in $[1, \alpha)$. If we color each edge according to the parity of m in the expression $c\alpha^m$ for its length, then no K' similar to K will be monochromatic.

By Ramsey's theorem, if K is the set of edges of a regular k-simplex, then for large enough n, depending on K and the number of colors r, any r-coloring of E^n has a monochromatic K' congruent to K. The minimum size of n is in general not known. It is known that $n \neq o\left(2^{\frac{k}{2}}\right)$ but we may be able to improve on this because this estimate is based only on properties of abstract sets, whereas in our case the geometry may help. The next simplest set K is the four sides of a unit square. For this set we haven't settled the question even in E^2, either for congruence or similarity.

If we restrict the kinds of colorings we allow, then we do get some positive results. We say that a *line coloring* is a coloring of edges so that any two collinear edges have the same color.

Let K_t be the set of edges (in the coordinate directions) of the unit $t \times t$ square lattice.

Theorem 23. *If the edges of E^3 are line colored with 2 colors, there is a K'_t similar to K_t which has all its edges the same color.*

Proof. We use the following fact. Let $k_1 r > 0$. Then for large enough $n \geqslant n(k, r)$ depending on k and r, and for any r-coloring of the points of an $n \times n$ square lattice of points (with distance d between adjacent points), there is a $k \times k$ square sublattice (with distance $d' \geqslant d$) with all points the same color (Grünwald quoted in [5]).

Now let the edges of E^3 be line colored with 2 colors. Consider a plane $H \subset E^3$, say the x, y-plane, and let each point p *of* H be colored according to the color of the line orthogonal to H through p. Then for arbitrarily large n there is a monochromatic $n \times n$ square lattice L in H. We can assume that $L = \{(x, y, 0 | x$ and y are integers between 0 and $n - 1\}$.

Let H' be the y, z-plane, and let L' be the $n \times n$ lattice $\{(0, y, z) | y, z$ are integers between 0 and $n - 1\}$. Let each point p of L' be colored according to the color of the line through p orthogonal to H'. Then if n is large enough, there is an $m \times m$ square sublattice L'' with all points the same color, where we take m to be $n(t, r)$. We may assume $L'' = \{(0, id, jd) | 0 \leqslant i, j \leqslant m - 1\}$ (for some d with $d(m - 1) < (n - 1)$).

Let H'' be the x, z-plane and let L''' be the lattice $\{(id, 0, jd) | 0 \leqslant \leqslant i, j \leqslant m - 1\}$. Coloring points of L''' according to the colors of the lines through the points orthogonal to H''', there is a lattice, say $L^{(iv)} = = \{(id', 0, jd') | 0 \leqslant i, j \leqslant t - 1\}$ with all points one color (for some d').

By definition of the colorings, the lattice $L^* = \{(id', jd', kd') | 0 \leqslant \leqslant i, j, k \leqslant t - 1\}$ has all lines in any coordinate direction a single color, depending on the direction. Since there are only 2 colors, two directions have the same color, say the x and y directions. Then $K'_t = = \{(id', jd', 0) | 0 \leqslant i, j \leqslant t - 1\}$ has all lines the same color, and K'_t is similar to K_t. This completes the proof.

This result clearly extends to higher dimensional lattices and more colors by the same kinds of argument. One could also extend get argument to coloring "faces" by "plane colorings" in an analogous way.

As a final example, we show that although we don't always get monochromatic triangles similar to a given one, there may be sets of triangles one member of which must occur monochromatically.

Theorem 24. *The edges of* E^2 *can be line colored with* 2 *colors so that no triangle with all angles at most* $90°$ *has all three edges the same color. On the other hand, for every line coloring of* E^2 *with* 2 *colors and every* $\epsilon > 0$ *some triangle with all angles less than* $90° + \epsilon$ *has all three edges the same color. For every* $\epsilon > 0$ *and every* 2*-coloring of the edges of* E^2, *some triangles with all angles at most* $108° + \epsilon$ *has all three edges the same color.*

Proof. Consider a line coloring of E^2 with 2 colors, and suppose all triangles with all angles at most $90°$ have edges of both colors. If any two perpendicular lines are the same color, say red, then all lines not parallel to either of these must be blue, a contradiction, since this would give blue $90°$ triangles. Thus we can assume all pairs of perpendicular lines have opposite colors.

This implies that all parallel lines have the same color, and the color of a line (or edge) is determined only by the angle it makes with the x-axis. Suppose two red lines L_1 and L_2 make angles θ_1, θ_2, with $0 < \theta_2 - \theta_1 < 90°$, and some blue line L_3 makes an angle θ_3 with $\theta_1 < \theta_3 < \theta_2$. Then L_1, L_2 and any L_3' perpendicular to L_3 and not through $L_1 \cap L_2$ makes an acute triangle with red edges, a contradiction. These observations imply that for some θ_1 and θ_2 with $|\theta_1 - \theta_2| = 90°$, all edges with angles in (θ_1, θ_2) are red, and all others blue. This establishes the first two statements.

For the third statement we consider a special set of eight points. Let p_1, p_2, p_3, p_4, p_5 be the vertices of a regular pentagon. Let q be the point where the extensions of the sides p_5, p_1 and p_3, p_4 meet. Let L be a line parallel to the side p_4, p_5 meeting the extended sides in points q_1 and q_5 lying respectively between p_5 and q, and p_4 and q. Choose L sufficiently close to q so that the angles formed by $q_1 p_3 q_5$ and $q_1 p_1 q_5$ are less than $\epsilon > 0$. Now let K be the set of edges given as follows: $p_i p_j$, $i, j = 1, 2, 3$; $p_i q_j$, $i = 1, 2, 3$, $j = 1, \dots, 5$; $q_i q_j$, $|i - j| \equiv 1 \pmod 5$. That is, K is the edge of a triangle and a pentagon and all edges connecting them. But this figure can be seen to have the property that for every 2-coloring of the edges there is a monochromatic triangle [4]. All triangles have angles of at most $108° + \epsilon$. This completes the proof.

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