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Ramsey's Theorem — A New Lower Bound

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This paper gives improved asymptotic lower bounds to the Ramsey function $R(k, t)$. Section 1 considers the symmetric case $k = t$ while the more general case is considered in Section 2.

1. THE SYMMETRIC CASE

Define $R(k)$ to be the minimal integer n so that if the edge of K_n (the complete graph on n points) are two colored there is a set S of k vertices such that all edges $\{x, y\}$, $x, y \in S$, are the same color.

The existance of $R(k)$ for all k is a special case of Ramsey's Theorem for which an enormous literature exists. The "standard" proof (see, e.g., [4]) of the existance of $R(k)$ yields

$$R(k) \leqslant \binom{2k-2}{k-1},$$

which has been slightly improved recently to

$$R(k) \leqslant \frac{c \log \log k}{\log k} \binom{2k-2}{k-1}$$

(Yackel [5]). It is expected that further small improvements could be made.

The lower bound on $R(k)$, due to Erdös [1], is generally considered the canonical example of the "probabilistic method" in combinatorial mathematics. We shall outline this proof, and then show how a new method of L. Lovász gives a slight improvement.

We let P denote probability, $\vee$ denote "or," concatenation or $\wedge$ denote "and," and bar denote negation. We state the following obvious lemma without proof.

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LEMMA 1. *Let A_i, $1 \leqslant i \leqslant m$, be events in a probability space with $P(A_i) \leqslant p$. If $mp < 1$ then*

$$P(\bar{A}_1 \cdots \bar{A}_m) > 0.$$

THEOREM 1 (Erdös [1]). *If*

$$\binom{n}{k} 2^{1-\binom{n}{k}} < 1, \tag{1}$$

then $R(k) \geqslant n$.

Proof. We call a coloring *good* if there is no set S of k vertices, all of whose edges are the same color.

Fix k, n satisying (1). Let $\mathbf{G}$ be a random 2-coloring of K_n. That is, each edge is independently colored, equally probably either color. If S is a set of k vertices, let A_S be the event that all edges on S are the same color. Clearly

$$P[A_S] = 2^{1-\binom{k}{2}}.$$

There are $\binom{n}{k}$ different S. Hence, given (1) and Lemma 1,

$$P\left[\wedge \bar{A}_S\right] > 0,$$

and therefore there is a G for which all A_S are false. That is, there is a good coloring of K_n. This implies $R(k) \geqslant n$.

An application of Sterling's Formula yields the following.

COROLLARY 1. $R(k) \geqslant k2^{k/2}[(1/e\sqrt{2}) + o(1)]$.

The improvement of Theorem 1 is based on the independence of A_S, A_T if $|S \cap T| \leqslant 1$. To make use of the "partial independence" of the A's we use the following elementary, but far reaching, result of L. Lovász.

LEMMA 2. (Lovász Local Theorem). *Let G be a finite graph with maximal degree d and vertices $1,\ldots, m$. Let A_i, $1 \leqslant i \leqslant m$ be events in a probability space such that A_i is independent of $\{A_j : (i,j) \in E(G)\}$. Assume $P(A_i) \leqslant p$ for $1 \leqslant i \leqslant m$. If $4dp < 1$ then*

$$P(\bar{A}_1 \cdots \bar{A}_m) > 0.$$

For completeness, we outline the proof given in [3]. We show, by induction on m,

$$P(A_1 \mid \bar{A}_2 \cdots \bar{A}_m) \leqslant 1/2d. \tag{2}$$

Assume A_1 independent of A_i, $i > d + 1$.

$$P(A_1 \mid \bar{A}_2 \cdots \bar{A}_m) = \frac{P(A_1\bar{A}_2 \cdots \bar{A}_{d+1} \mid \bar{A}_{d+2} \cdots \bar{A}_m)}{P(\bar{A}_2 \cdots \bar{A}_{d+1} \mid \bar{A}_{d+2} \cdots \bar{A}_m)} \tag{3}$$

We prove (2) by bounding numerator and denominator of (3). The numerator

$$P(A_1\bar{A}_2 \cdots \bar{A}_{d+1} \mid \bar{A}_{d+2} \cdots \bar{A}_m) \leqslant P(A_1 \mid \bar{A}_{d+2} \cdots \bar{A}_m) = P(A_1) \leqslant \tfrac{1}{4}d.$$

The denominator

$$P(\bar{A}_2 \cdots \bar{A}_{d+1} \mid \bar{A}_{d+2} \cdots \bar{A}_m) \geqslant 1 - \sum_{i=2}^{d+1} P(A_i \mid \bar{A}_{d+2} \cdots \bar{A}_m)$$
$$\geqslant 1 - d(\tfrac{1}{2}d) \geqslant \tfrac{1}{2},$$

where the penultimate inequality has required the inductive hypothesis.

THEOREM 2. *If*

$$4\binom{k}{2}\binom{n}{k-2} 2^{1-\binom{k}{2}} < 1, \tag{4}$$

then $R(k) \geqslant n$.

Proof. Let A_S be as in Theorem 1. Then A_S is independent of $\{A_T : |S \cap T| \leqslant 1\}$ since S shares no edges with the T's. We apply the Lovász Local Theorem with

$$d = |\{T : |T| = k, |S \cap T| > 1\}| \leqslant \binom{k}{2}\binom{n}{k-2}.$$

COROLLARY 2. $R(k) \geqslant k2^{k/2}[(\sqrt{2}/e) + o(1)]$.

The corollary is again a simple application of Sterling's Formula. This improvement of the lower bound by a factor of 2 does not lessen the gap between the bounds in any significant way. It is, however, the first improvement in the lower bound of $R(k)$ in 27 years.

If n is picked slightly less than the critical value in (2) then

$$\binom{n}{k} 2^{1-\binom{k}{2}} \ll 1.$$

Thus, not only does there exist a two coloring of K_n but "almost all" such colorings are good. However, using the Lovász Local Theorem, $P(\bar{A}_1 \cdots \bar{A}_m)$ may be very small. One can show that most colorings on

$k2^{k/2}(\sqrt{2}/e + o(1))$ points are not good. We have found a "rare" good coloring with a "random" method.

2. The General Case

Define $R(k, t)$ to be the minimal integer n so that if the edges of K_n are colored Red and Blue there is either a set S of k vertices all of whose edges are Red or a set T of t vertices all of whose edges are Blue.

The "standard" proof [13] of Ramsey's Theorem yields

$$R(k, t) \leqslant \binom{k+t-2}{k-1}. \tag{5}$$

We shall focus our attention on the case k fixed, $t \to \infty$. Then

$$R(k, t) \leqslant c_k t^{k-1},$$

where c_k is a constant dependent on k. This result has been slightly improved to

$$R(k, t) \leqslant c \left(\frac{\log \log t}{\log t}\right) t^{k-1}$$

(Yackel [4]).

We first derive a lower bound for $R(k, t)$ by generalizing Theorem 1.

Theorem 3. *If there exists p, $0 \leqslant p \leqslant 1$, such that*

$$\binom{n}{p} p^{\binom{k}{2}} + \binom{n}{t} (1-p)^{\binom{t}{2}} < 1, \tag{6}$$

then $R(k, t) > n$.

Proof. Fix k, t, p, n satisfying (6). Let G be a two coloring (Red and Blue) of K_n where each edge is colored Red with probability p and these probabilities are mutually independent. If S is a set of k vertices let A_S be the event that all edges on S are Red. If T is a set of t vertices let B_T be the event that all edges in T are blue. Clearly,

$$P[A_S] = p^{\binom{k}{2}} \qquad P[B_T] = (1-p)^{\binom{t}{2}}.$$

So, given (5) and Lemma 1,

$$P[\Lambda \bar{A}_S \wedge \Lambda \bar{B}_T] > 0, \tag{7}$$

and thus $R(k, t) > n$.

Now let us fix k, let $t \to \infty$, and consider the asymptotic consequences of (6). By selecting $p = n^{-2/(k-1)}$, we get

$$\binom{n}{k} p^{\binom{k}{2}} < n^k p^{k(k-1)/2}/k! = 1/k!$$

Using the inequality $1 - p < e^{-p}$, we have

$$\begin{aligned}\binom{n}{t} (1-p)^{\binom{t}{2}} &< (n^t/t!)\, e^{-pt(t-1)/2} \\ &< [ne^{-p(t-1)/2}]^t/t!. \qquad (8)\end{aligned}$$

If $t - 1 \geqslant (2 \ln n)/p$, then (6) holds. Asymptotically, then,

$$R\,(k, (2 \ln n)\, n^{2/(k-1)}(1 + o(1))) > n \qquad (9)$$

Expressing (9) in terms of the parameter t, we have the following.

COROLLARY 3. *For k fixed, $t \to \infty$*

$$R(k, t) > t^{(k-1)/2+o(1)}. \qquad (10)$$

A major open problem in this area is to determine $\alpha = \alpha(k)$ such that $R(k, t) = t^{\alpha+o(1)}$. It is not known if such an α exists. For $k = 3$ Erdös [2] has shown

$$R(3, t) > ct^2/(\ln t)^2, \qquad (11)$$

and hence $\alpha(3) = 2$. A plausible conjecture is that $\alpha(k) = k - 1$ for all $k \geqslant 3$ but this is not even known for $k = 4$.

We now give a generalization of the Lovász Local Theorem.

THEOREM 4. *Let G be a finite graph on vertices $1,\ldots, m$. Let A_i, $1 \leqslant i \leqslant m$, be events in a probability space such that A_i is independent of $\{A_j : \{i, j\} \in E(G)\}$. For $1 \leqslant i \leqslant m$ assume*

$$\sum_{\{i,j\}\in G} P(A_j) < \tfrac{1}{4} \qquad (12)$$

Assume, further, that $P(A_j) < 1$ for all j. Then

$$P(\bar{A}_1 \cdots \bar{A}_m) > 0. \qquad (13)$$

When all $P(A_i)$ are equal the statement of Theorem 4 reduces to Lemma 2. The proof will parallel that of Lemma 2. We first observe that if $P(A_j) > \frac{1}{4}$ then, by (12), A_j is mutually independent of the other A's and hence it suffices to show (13) with A_j deleted. We therefore assume $P(A_j) \leqslant \frac{1}{4}$ for all j.

We show, by induction on m,

$$P(A_1 \mid \bar{A}_2 \cdots \bar{A}_m) \leqslant 2P(A_1) \tag{14}$$

Assume 1 is adjacent to 2, 3,..., d in G. Then

$$P(A_1 \mid \bar{A}_2 \cdots \bar{A}_m) = \frac{P(A_1\bar{A}_2 \cdots \bar{A}_d \mid \bar{A}_{d+1} \cdots \bar{A}_n)}{P(\bar{A}_2 \cdots \bar{A}_d \mid \bar{A}_{d+1} \cdots \bar{A}_n)}. \tag{15}$$

The numerator of (15) is $\leqslant P(A_1)$ as before. The denominator

$$\begin{aligned} P(\bar{A}_2 \cdots \bar{A}_d \mid \bar{A}_{d+1} \cdots \bar{A}_n) &\geqslant 1 - \sum_{i=2}^{d} P(A_i \mid \bar{A}_{d+1} \cdots \bar{A}_n) \\ &\geqslant 1 - \sum_{i=2}^{d} 2P(A_i) \\ &\geqslant \tfrac{1}{2}, \end{aligned} \tag{16}$$

where the penultimate inequality has required the inductive hypothesis. Finally

$$\begin{aligned} P(\bar{A}_1 \cdots \bar{A}_m) &= \prod_{i=1}^{m} P(\bar{A}_i \mid \bar{A}_{i+1} \cdots \bar{A}_m) \\ &\geqslant \prod_{i=1}^{m} (1 - 2P(A_i)) \\ &> 0. \end{aligned} \tag{17}$$

This completes the proof.

We apply Theorem 4 to improve the lower bound for $R(k, t)$. We first give the precise result. Let $A \subseteq \{1,..., n\}$, $| A | = a$. Denote by $f(a, b, n)$ the number of $B \subseteq \{1,..., n\}$, $| B | = b$, such that $| A \cap B | \geqslant 2$.

THEOREM 5. *Let $k \leqslant t \leqslant n$. If there exists p, $0 < p < 1$, so that*

$$f(t, k, n)\, p^{\binom{k}{2}} + f(t, t, n)(1 - p)^{\binom{t}{2}} < \tfrac{1}{4}, \tag{18}$$

then $R(k, t) > n$.

Proof. Let **G** be as in Theorem 3. The events A_S, B_T satisfy (6) and we need show (7). Our assumption (18) states that each B_T is independent of all events except those with total probability $< \frac{1}{4}$. An event A_S is independent of even more events, since $k \leqslant t$. (That is, $f(a, b, n)$ is monotone increasing in a). Therefore the conditions of Theorem 4 are met, implying (7).

Now we examine the asymptotic consequences of Theorem 5 in the case of k fixed, $t \to \infty$. We bound

$$f(t, t, n) \leqslant \binom{n}{t} \leqslant n^t,$$

$$f(t, k, n) \leqslant \binom{t}{2}\binom{n}{k-2} = c_k t^2 n^{k-2}.$$

If there exists p, $0 < p < 1$ so that

$$c_k t^2 n^{k-2} p^{\binom{k}{2}} < \tfrac{1}{8}$$

and

$$\binom{n}{t}(1-p)^{\binom{t}{2}} < \tfrac{1}{8},$$

then (18) holds. Set $\beta = (k-2)/(\binom{k}{2} - 2)$ and, for any $0 < \delta < \epsilon$, $t = n^{\beta+\epsilon}$ and $p = n^{-\epsilon-\beta+\delta}$. For n sufficiently large (18) holds and thus $R(k, t) > n$. Expressing n in terms of t.

Corollary 3. *For k fixed, $t \to \infty$*

$$R(k, t) > t^{\alpha+o(1)}$$

where

$$\alpha = \left(\binom{k}{2} - 2\right)\Big/(k-2).$$

Note that for k large $\alpha \sim (k+1)/2 + o(1)$. Table 1 gives the various upper and lower bounds for $\alpha(k)$

TABLE I

Bounds on $\alpha(k)$

k	Lower by Corollary 3	Lower by Theorem 5	Lower by Erdös [2]	Upper by (5)
3	1	1	2	2
4	$1\frac{1}{2}$	2	2	3
5	2	$2\frac{2}{3}$	2	4
6	$2\frac{1}{2}$	$3\frac{1}{4}$	2	5
7	3	3.8	2	6

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