Small Ramsey Numbers

Exposition by William Gasarch

October 27, 2024

Lets Party Like Its January of 2019

Recall the first theorem one usually hears in Ramsey Theory and can tell your non-math friends about.

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We define graphs and complete graphs and state this theorem in those terms.

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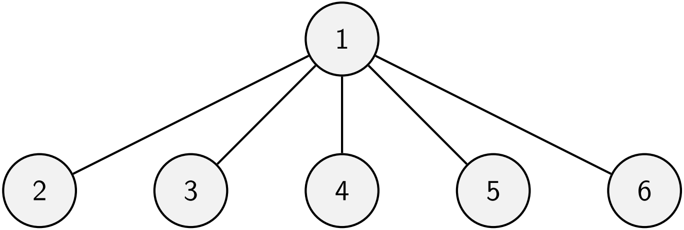

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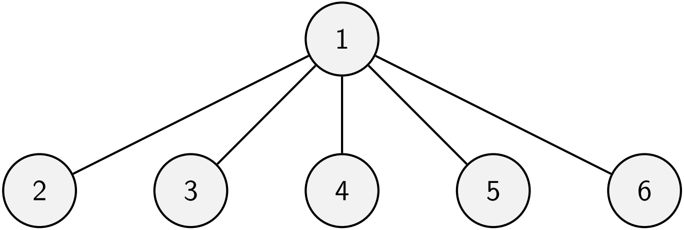

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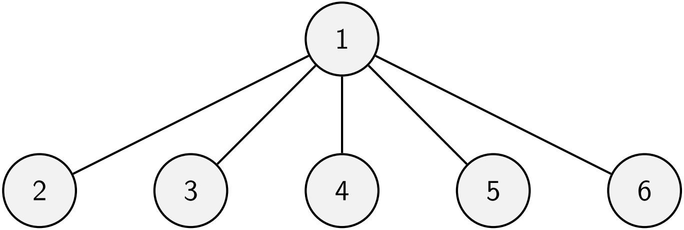

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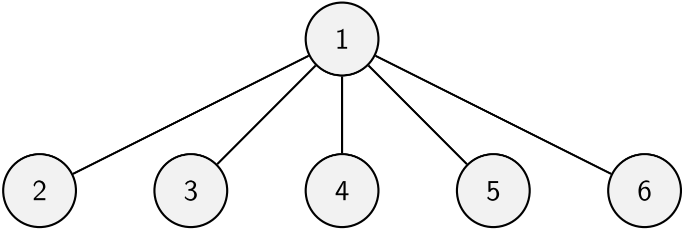

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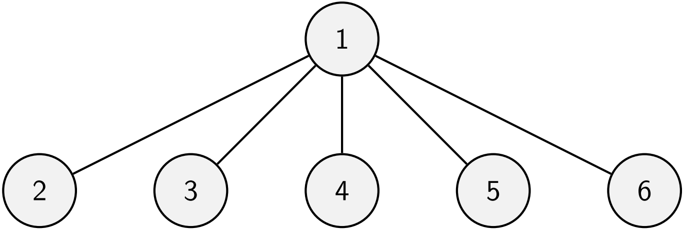

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Def The **degree (deg)** of a vertex is how many edges use it.
In the above graph $\deg(1) = 5$ and
$\deg(2) = \deg(3) = \deg(4) = \deg(5) = \deg(6) = 1$.

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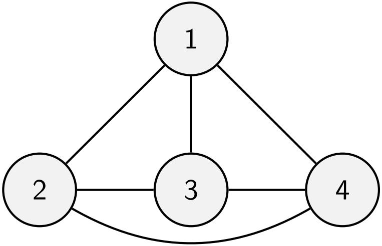

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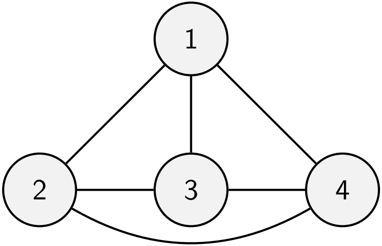

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Note Every vertex of K_n has degree $n - 1$.

More Notation

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- $\exists$ means **there exists**
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   Bill Gasarch and the Red Cliques!

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We prove this in the next few slides.

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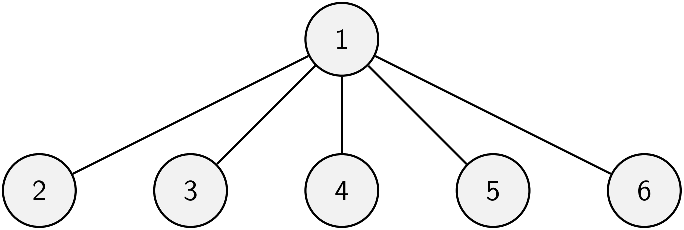

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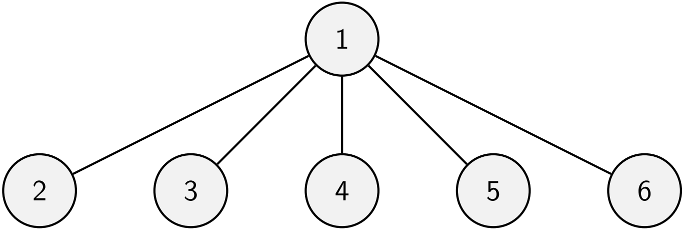

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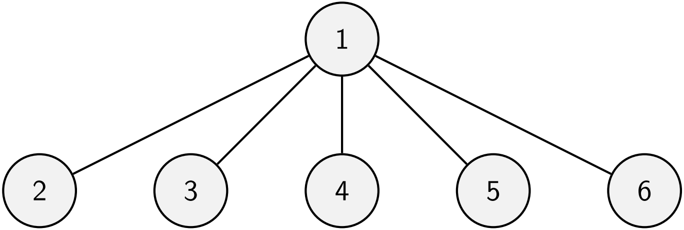

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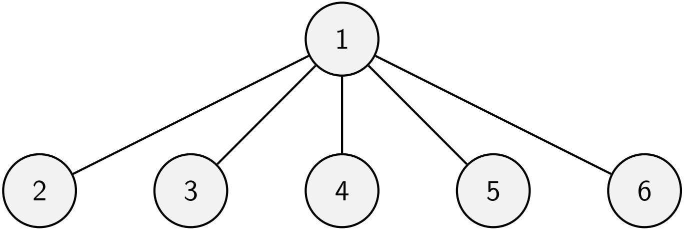

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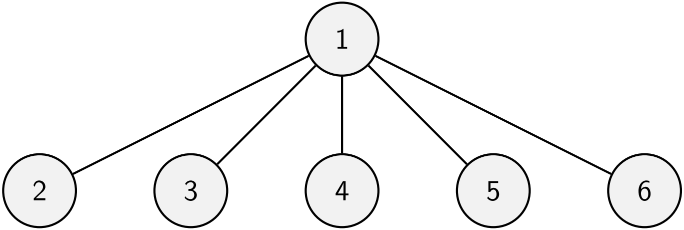
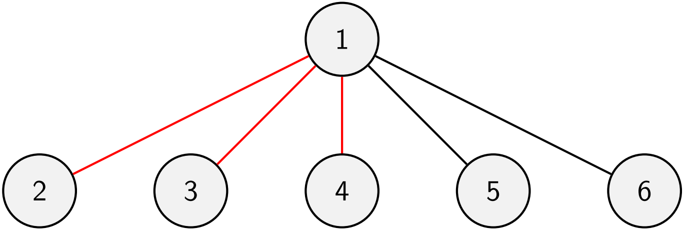
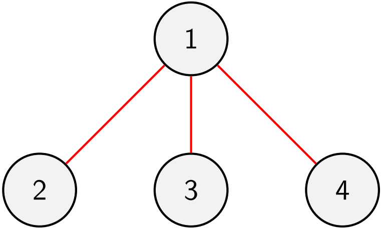


There are 5 edges coming out of vertex 1.
They are 2 colored.
$\exists$ 3 edges from vertex 1 that are the same color.
We can assume $(1,2)$, $(1,3)$, $(1,4)$ are all **RED**.

(1,2), (1,3), (1,4) are **RED**



We Look Just at Vertices 1,2,3,4



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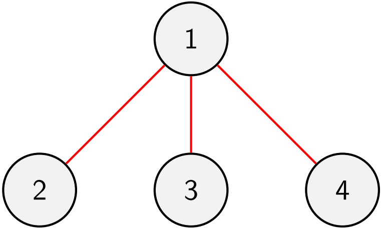

If $(2,3)$ is **RED** then get **RED** Triangle. So assume (2,3) is **BLUE**.

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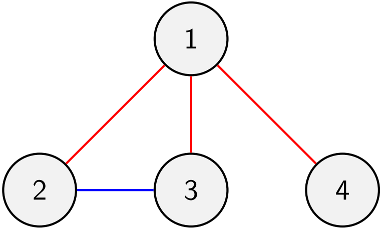

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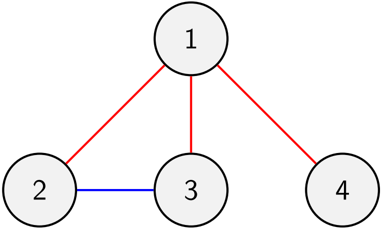

If (3,4) is **RED** then get **RED** triangle. So assume (3,4) is **BLUE**.

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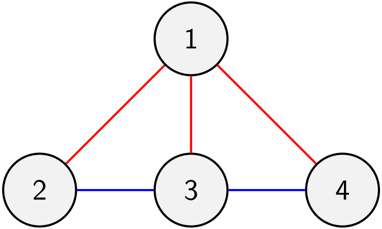

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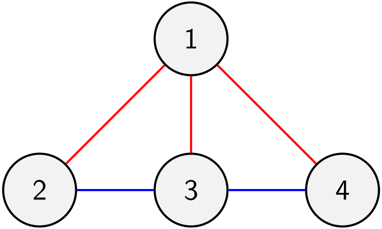

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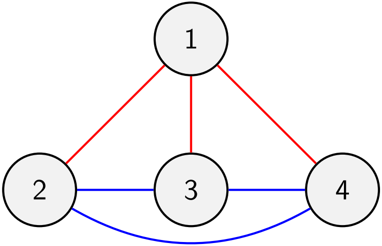

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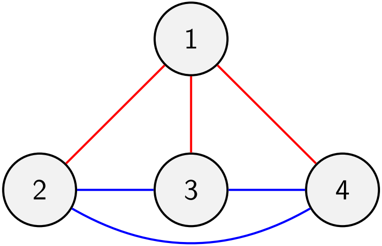

Note that there is a **BLUE** triangle with verts $2, 3, 4$. Done!

What if we color edges of K_5?

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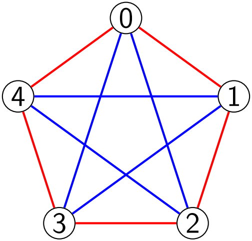

This graph is not arbitrary.
$SQ_5 = \{x^2 \pmod 5 : 0 \le x \le 4\} = \{0, 1, 4\}$.

- If $i - j \in SQ_5$ then **RED**.
- If $i - j \notin SQ_5$ then **BLUE**.

Asymmetric Ramsey Numbers

Definition $R(a,b)$ is least n such that for all 2-colorings of K_n there is **either** a red K_a or a blue K_b.

1. $R(a,b) = R(b,a)$.
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Proof left to the reader, but its easy.

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1. There is a vertex with large **Red** Deg.
2. There is a vertex with large **Blue** Deg.
3. All verts have small **Red** degree and small **Blue** degree.

Some Vertex v Has Large Red Deg

Case 1 $(\exists v)[\deg_R(v) \geq R(a-1,b)]$.

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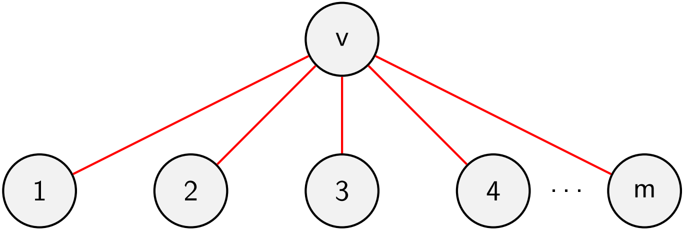

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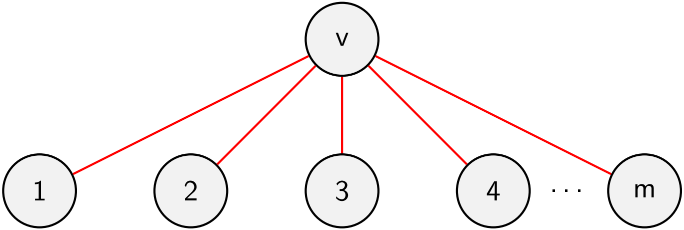

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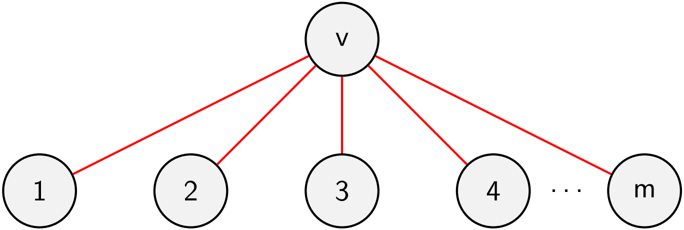

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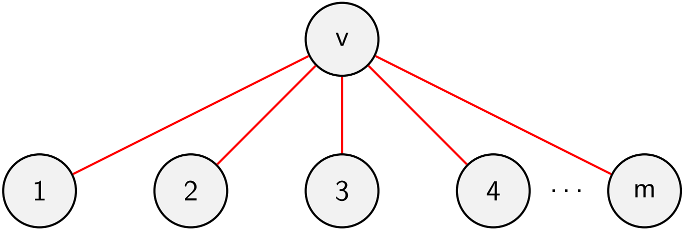

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Case 1.2 There is a **Blue** K_b in $\{1,\ldots,m\}$. DONE.
Case 1.3 Neither. **Impossible** since $m = R(a-1,b)$.

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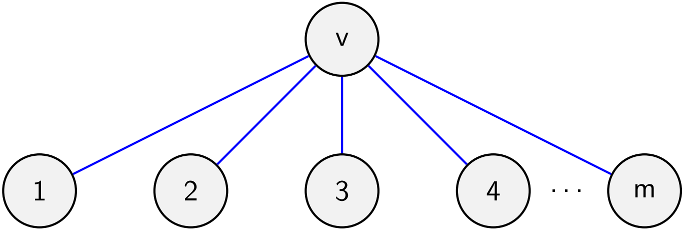

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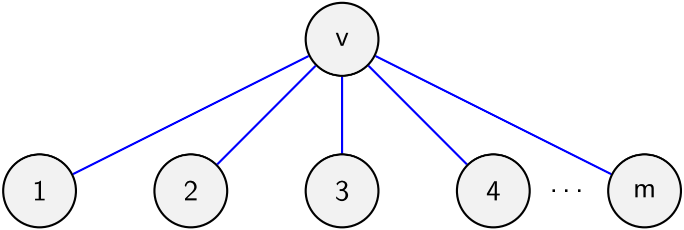

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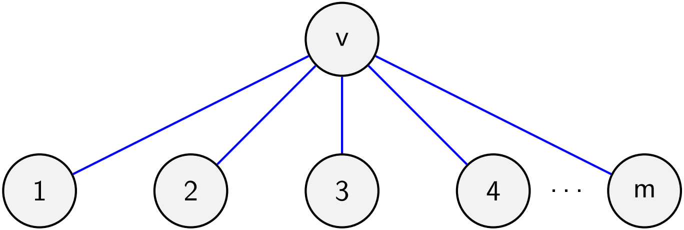

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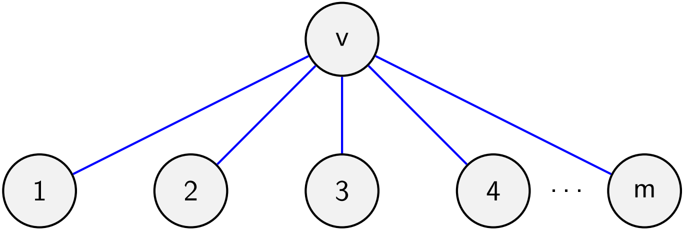

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All Verts: Small Red Deg and Small Blue Deg

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2. $(\forall v)[\mathrm{deg}_B(v) \le R(a,b-1)-1]$

Hence

$$(\forall v)[\mathrm{deg}(v) \le R(a-1,b)+R(a,b-1)-2 = n-2]$$

All Verts: Small Red Deg and Small Blue Deg

Case 3 Negate Case 1 and Case 2:

1. $(\forall v)[\mathrm{deg}_R(v) \le R(a-1,b)-1]$ and
2. $(\forall v)[\mathrm{deg}_B(v) \le R(a,b-1)-1]$

Hence

$$(\forall v)[\mathrm{deg}(v) \le R(a-1,b)+R(a,b-1)-2 = n-2]$$

Not possible since every vertex of K_n has degree $n-1$.

Lets Compute Bounds on $R(a, b)$

- $R(3,3) \leq R(2,3) + R(3,2) \leq 3 + 3 = 6$
- $R(3,4) \leq R(2,4) + R(3,3) \leq 4 + 6 = 10$
- $R(3,5) \leq R(2,5) + R(3,4) \leq 5 + 10 = 15$
- $R(3,6) \leq R(2,6) + R(3,5) \leq 6 + 15 = 21$
- $R(3,7) \leq R(2,7) + R(3,6) \leq 7 + 21 = 28$
- $R(4,4) \leq R(3,4) + R(4,3) \leq 10 + 10 = 20$
- $R(4,5) \leq R(3,5) + R(4,4) \leq 15 + 20 = 35$
- $R(5,5) \leq R(4,5) + R(5,4) \leq 35 + 35 = 70.$

Table of Bounds

$R(a,b)$	Bound on $R(a,b)$
$R(3,3)$	6
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Can we make some improvements to this?

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Can we make some improvements to this? YES!
We need a theorem.

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Can we make some improvements to this? YES!
We need a theorem. We first do an example.

A Graph on 9 Vertices with all verts Deg 3?

Thm There is NO graph on 9 verts, with every vertex of deg 3.

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We generalize this on the next slide.

Handshake Lemma

Lemma Let $G = (V, E)$ be a graph.

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$$\sum_{v \in V_{\mathrm{even}}} \deg(v) + \sum_{v \in V_{\mathrm{odd}}} \deg(v) = \sum_{v \in V} \deg(v) = 2|E| \equiv 0 \pmod 2.$$

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Handshake Lemma If all pairs of people in a room shake hands, even number of shakes. (Pre-COVID when people shook hands.)

Corollary of Handshake Lemma

Impossible to have a graph on an odd number of verts where every vertex is of odd degree.

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And NOW to our improvements on small Ramsey numbers.

$R(3, 4) \leq 9$ Case 1

Assume we have a 2-coloring of the edges of K_9.

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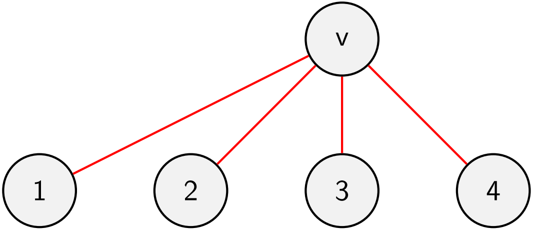

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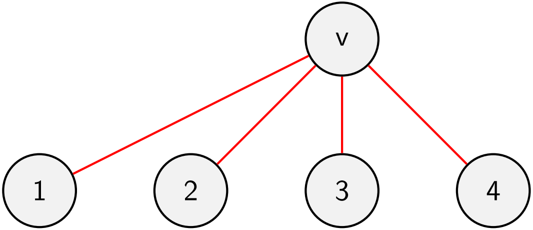

1) If **any** of $\{1,2\}$,$\{1,3\}$, $\{1,4\}$, $\{2,3\}$, $\{2,4\}$, $\{3,4\}$ are **RED**, have **RED** K_3.

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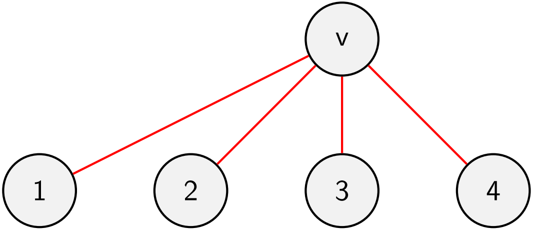

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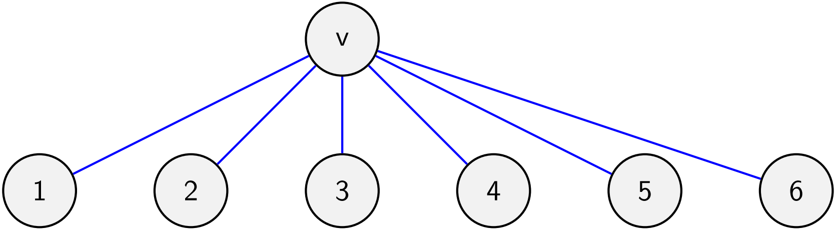
2) If **all** of $\{1,2\}$,$\{1,3\}$, $\{1,4\}$, $\{2,3\}$, $\{2,4\}$, $\{3,4\}$ are **BLUE**, have **BLUE** K_4.

$R(3,4) \le 9$ Case 2

Case 2 $(\exists v)[\deg_R(v) \le 2]$, so $\deg_B(v) \ge 6$.



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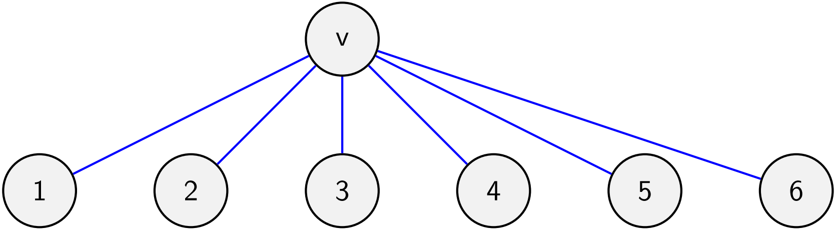

(1) There is a **RED** K_3 in $\{1, 2, 3, 4, 5, 6\}$. Have **RED** K_3.

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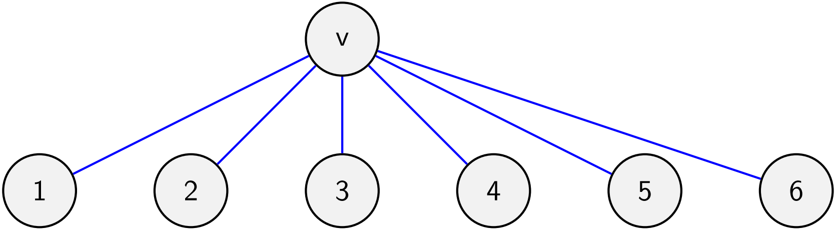

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$R(3,4) \leq 9$ Case 3

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SO the **RED** graph is a graph on 9 verts with all verts of degree 3.
This is impossible!

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Theorem $R(a,b) \le$

1. $R(a,b-1) + R(a-1,b)$ always.
2. $R(a,b-1) + R(a-1,b) - 1$ if
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Proof left to the Reader.

Some Better Upper Bounds

- $R(3,3) \leq R(2,3) + R(3,2) \leq 3 + 3 = 6.$
- $R(3,4) \leq R(2,4) + R(3,3) \leq 4 + 6 - 1 = 9.$
- $R(3,5) \leq R(2,5) + R(3,4) \leq 5 + 9 = 14.$
- $R(3,6) \leq R(2,6) + R(3,5) \leq 6 + 14 - 1 = 19.$
- $R(3,7) \leq R(2,7) + R(3,6) \leq 7 + 19 = 26$
- $R(4,4) \leq R(3,4) + R(4,3) \leq 9 + 9 = 18.$
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Are these tight? Some yes, some no.

$R(3,3) \geq 6$

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$COL(a,b) =$ **RED** if $a - b \equiv SQ \pmod 5$, **BLUE** OW.

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$COL(a,b) =$ **RED** if $a - b \equiv SQ \pmod 5$, **BLUE** OW.

Note $-1 = 2^2 \pmod 5$. Hence $a - b \in SQ$ iff $b - a \in SQ$. So the coloring is well defined.

$R(3,3) \geq 6$

$COL(a,b) =$ **RED** if $a-b \equiv SQ \pmod{5}$, **BLUE** OW.

- Squares mod 5: 1,4.
- If there is a **RED** triangle then $a-b$, $b-c$, $c-a$ all SQ's. SUM is 0. So

$$x^2+y^2+z^2 \equiv 0 \pmod{5} \text{ Can show impossible}$$

- If there is a **BLUE** triangle then $a-b$, $b-c$, $c-a$ all non-SQ's. Product of nonsq's is a sq. So $2(a-b)$, $2(b-c)$, $2(c-a)$ all squares. SUM to zero- same proof.

UPSHOT $R(3,3)=6$ and the coloring used math of interest!

$R(4,4) = 18$

$R(4,4) \geq 18$: Need coloring of K_{17} w/o mono K_4.

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Use
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Same idea as above for K_5, but more cases.
UPSHOT $R(4,4) = 18$ and the coloring used math of interest!

$R(3,5) = 14$

$R(3,5) \geq 14$: Need coloring of K_{13} w/o **RED** K_3 or **BLUE** K_5.

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Use
$COL(a,b) =$ **RED** if $a - b \equiv CUBE \pmod{14}$, **BLUE** OW.

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Vertices are $\{0,\ldots,13\}$.

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$COL(a,b) =$ **RED** if $a - b \equiv CUBE \pmod{14}$, **BLUE** OW.

Same idea as above for K_5, but more cases.

UPSHOT $R(3,5) = 14$ and the coloring used math of interest!

$R(3,4) = 9$

This is a subgraph of the $R(3,5)$ graph

$R(3,4) = 9$

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UPSHOT $R(3,4) = 9$ and the coloring used math of interest!

Can we extend these Patterns?

Good news $R(4,5) = 25$.

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THATS IT.

Can we extend these Patterns?

Good news $R(4,5) = 25$.

Bad news

THATS IT.

No other $R(a, b)$ are known using NICE methods.

Summary of Bounds

$R(a,b)$	Old Bound	New Bound	Opt	Int?
$R(3,3)$	6	6	6	Y
$R(3,4)$	10	9	9	Y
$R(3,5)$	15	14	14	Y
$R(3,6)$	21	19	18	Lower-Y
$R(3,7)$	28	27	23	Lower-Y
$R(4,4)$	20	18	18	Y
$R(4,5)$	35	31	25	N
$R(5,5)$	70	62	??	??

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$R(5,5)$: $43 \leq R(5,5) \leq 49$. So far not mathematically interesting.

Moral of the Story

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   (Joel Spencer) The Law of Small Numbers: Patterns that persist for small numbers will vanish when the calculations get to hard.
2. Seemed like a nice **Math** problem that would involve interesting and perhaps deep mathematics. No. The work on it is interesting and clever, but (1) the math is not deep, and (2) progress is slow.

When Will We Know $R(5,5)$

1. (Quote from Joel Spencer): *Erdos asks us to imagine an alien force, vastly more powerful than us, landing on Earth and demanding the value of* $R(5,5)$ *or they will destroy our planet. In that case, he claims, we should marshal all our computers and all our mathematicians and attempt to find the value. But suppose, instead, that they ask for* $R(6,6)$. *In that case, he believes, we should attempt to destroy the aliens.*

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