A Survey of Recursive Combinatorics

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1 Introduction

Many theorems in infinite combinatorics have noneffective proofs. Nerode's recursive mathematics program [124] involves looking at noneffective proofs and seeing if they can be made effective. The framework is recursion-theoretic. For example, to see if the theorem (which we denote T) 'Every vector space has a basis' has an effective proof, one might look at the statement (which we denote S) 'Given a recursive vector space, one can effectively find a basis for it'. If statement S is false, then there can be no effective proof of Theorem T (statement S is false, see [123]). We examine theorems about infinite combinatorics in this context. Given a theorem in infinite combinatorics that has a noneffective proof, we ask the following three questions:

1. Is the recursive analogue true? (Usually no.)
2. Is some modification of the recursive analogue true? (Usually yes.) The modification can be either
   (a) recursion-theoretic (i.e., in Theorem T above we might replace 'one can effectively find' with 'one can find recursively in the oracle A' for some well behaved A), or
   (b) combinatorial (i.e., change the type of object you want to find).
3. How hard is it (in the arithmetic hierarchy) to determine if a given instance of the theorem has a recursive solution?

Item i is an example of recursive mathematics. This field has its roots in two early papers in recursive algebra [56, 135]; however, Nerode is the modern founder of recursive mathematics [124]. Item ii.a is an attempt to measure just how noneffective the proofs are, and is evident in the work of Jockusch on Ramsey's Theorem [87]. Item ii.b is an attempt to recover the effective aspects of combinatorics in infinite domains, and was first mentioned by Kierstead in his work on Dilworth's theorem for infinite partial orders [94]. Item iii was an outgrowth of an attempt to link recursive graph theory to complexity theory. For example, 3-colorability of finite graphs is of unknown complexity (since it is NP-complete), so the problem of determining if an infinite graph is 3-colorable might be a good analogue. This was the (unstated) motivation behind the first paper that analyzed such issues

[13]. Item iii was first pursued by Beigel and Gasarch [13]. Subsequent work has been done by Harel [77], and Gasarch and Martin [66].

There are not many published papers pursuing item iii, so many such results appear here for the first time.

1.1 General Philosophy

In each section of this paper we will state a noneffective theorem, sketch a proof, and then consider possible recursive analogues and their modifications. More detail than usual will be given in the proofs of the noneffective theorems. This is because (1) if we want to examine a noneffective proof, then that proof ought to be in this paper, and (2) these proofs tend to not get written down, as most authors (rightly so) just say 'by the usual compactness arguments' or 'by König's lemma on infinite trees.'

Some of the proofs in this paper look similar and can probably be put into an abstract framework. Indeed, (1) abstract frameworks for constructions of recursive partial orders [97] and recursive graphs [31] have been worked out, and (2) an abstract framework for recursion-theoretic theorems, namely the theory of Π_1^0 classes [35] has been worked out. We do not use these or other frameworks, because such devices make reading more difficult for readers not familiar with the area.

Each section has a subsection of miscellaneous results, as does the paper. The results mentioned here are not proven and are intended more to point the reader to references.

It was my intention to mention every single result in recursive combinatorics that was known. While it is doubtful that I've succeeded, I believe I have come close.

1.2 Related Work

Downey has written a survey [49] of recursive linear orderings. In addition, the last chapter of Rosenstein's book [143] is on recursive linear orderings. Cenzer and Remmel have written a survey [35] on Π_0^1 classes. These arise often in recursive combinatorics and in other parts of recursive mathematics.

In this survey we often give index-set results about how hard certain combinatorical results are. Cenzer and Remmel [34] have refined some of those results.

2 Definitions and Notation

In this section we present definitions and notations that are used throughout this paper. We do not define notions relevant to combinatorical objects in this section, but rather in the section where they are used (e.g., 'homogeneous set' is defined in the section on Ramsey Theory).

All recursion-theoretic notation is standard and follows [159].

Notation 2.1 Let $A \subseteq \mathsf{N}$ and $k \in \mathsf{N}$.

1. $[A]^k$ is the set of all k-element subsets of A.
2. $\exists^\infty$ means 'for infinitely many'.
3. $\forall^\infty$ means 'for all but a finite number of'.
4. $\mu x[P(x)]$ means 'the least x such that $P(x)$ holds'.
5. $A()$ and χ_A both represent the characteristic function of A.
6. $\langle x, y\rangle$ is a recursive bijection from $\mathsf{N} \times \mathsf{N}$ to N. Similarly for $\langle x_1, \ldots, x_n\rangle$. All functions in this paper are on 1 variable; however we may write $f(\langle x, y\rangle)$ and $\{e\}(\langle x, y\rangle)$ as $f(x, y)$ and $\{e\}(x, y)$.
7. $\{0\}$, $\{1\}$, $\ldots$ is a standard list of all Turing machines. We use $\{e\}$ for both the Turing machine and the partial recursive function that it computes. W_e is the domain of $\{e\}$.
8. $\{0\}^{()}$, $\{1\}^{()}$, $\ldots$ is a standard list of all oracle Turing machines. We use $\{e\}^A$ for both the oracle Turing machine with oracle A, and the partial recursive-in-A function that it computes. W_e^A is the domain of $\{e\}^A$.
9. K is the halting set, K_s denotes the first s elements in some fixed recursive enumeration of K. $TOT01$ denotes the set of indices of total functions that are 0-1 valued. $TOTINF01$ denotes the set of indices of total functions that are 0-1 valued and take on the value 0 infinitely often TOT_a is the set of indices of total functions whose image is contained in $\{1, \ldots, a\}$.

10. If A is a set then A', the *jump of A,* is $\{e : \{e\}^A(e)\downarrow\}$, i.e., the halting set relative to A. If f is a function, then f', the *jump of f*, is $\{e : \{e\}^f(e)\downarrow\}$, where $\{e\}^{()}$ is defined in such a way as to be able to access a function instead of a set.

11. If σ is a finite sequence of natural numbers then we think of it as being a function from $\{0, 1, \ldots, |\sigma| - 1\}$ to N, which we also denote by σ. The value of $\sigma(i)$ is the $i+1^{\mathrm{st}}$ element of the sequence (we do this since there is no '0^{th} element of a sequence'). Hence the value of $\sigma(i)$ is the value of the function at i.

12. If σ is an infinite sequence of natural numbers then we think of it as being a function from N to N, and the same conventions as in the last item apply.

13. If σ is a finite sequence and τ is a finite or infinite sequence then $\sigma \preceq \tau$ means that σ is a prefix of τ, and $\sigma \prec \tau$ means that σ is a proper prefix of τ.

14. If Σ is a set then Σ^* is the set of finite strings over Σ. The most usual uses are $\{0, 1\}^*$ and N^*.

15. λ denotes the empty string.

In this paper we often deal with functions where we care about the answer only if the input is of a certain form. We will ignore what happens otherwise. For example, in Section 5 we will have a convention by which certain numbers represent graphs, and others do not; and we may want $f(e)$ to be meaningful only if e represents a graph. For this reason we introduce the notion of a *Promise Problem*, originally defined (in a complexity theory context) in [53].

Definition 2.2 A *promise problem* is a set D together with a partial function f such that D is a subset of the domain of f. Let (D, f) be a promise problem, and X be a set. A *solution to* (D, f) is a total function g that agrees with f on D. (D, f) *is recursive in* X if there is a solution g to (D, f) such that $g \leq_{\mathrm{T}} X$. X *is recursive in* (D, f) if, for every solution g to (D, f), $X \leq_{\mathrm{T}} g$.

Definition 2.3 Let (D, A) be a promise problem where A is a 0-1 valued partial function. A *solution* to (D, A) is a *set* B that agrees with A on D. (D, A) *is* Σ_n if *some* solution is a Σ_n set. (D, A) *is* Σ_n*-hard* if *all* solutions are Σ_n-hard sets. A promise problem is Σ_n-complete if it is both in Σ_n and is Σ_n-hard. The same definitions apply to Π_n.

3 König's Lemma

König's lemma is an important theorem in infinite combinatorics. Many theorems in infinite combinatorics can be derived from it, including Theorems 4.3, 5.3, 6.5, and 7.3. We do not prove these theorems using König's lemma, since we want each chapter to be self contained.

In this section we (1) present a proof of the classical König's lemma, (2) show that a recursive analogue is false and give an index set version, and (3) state (but do not prove) a recursion-theoretic analogue that is true. The latter is due to Jockusch and Soare [90] and is called the 'low basis theorem.' We will use it in later chapters to obtain recursion-theoretic versions of theorems in infinite combinatorics.

3.1 Definition and Classical Version

Definition 3.1 A *tree* is a subset T of N^* such that if $\sigma \in T$ and $\tau \preceq \sigma$ then $\tau \in T$. A tree is *bounded* if there exists a function g such that, for all $\sigma \in T$, $|\sigma| \geq n+1 \Rightarrow \sigma(n) \leq g(n)$. (Recall that $\sigma(n)$ is the $(n+1)$st element of the sequence σ.)

Definition 3.2 Let T be a tree. An *infinite branch of* T is an infinite sequence σ such that every finite prefix of σ is in T. To each infinite branch σ we associate a function as indicated in Notation 2.1.xii. We view T as a set of functions by identifying T with the set of infinite branches of T.

Theorem 3.3 (König's Lemma [106]) *If T is an infinite tree and T is bounded then T has an infinite branch.*

Proof:

Let $\sigma_0 = \lambda$ and note that $\sigma_0 \in T$. Assume inductively that $\sigma_n \in T$ and the set $\{\sigma \in T : \sigma_n \preceq \sigma\}$ is infinite. Let

$$\begin{aligned} a_n = & \quad \mu x[\sigma_n x \in T \wedge |\{\sigma \in T : \sigma_n x \preceq \sigma\}| = \aleph_0] \\ \sigma_{n+1} & = \sigma_n x \end{aligned}$$

For all n a_n exists because T is bounded. The sequence $a_0, a_1, \ldots$ is an infinite branch of T. ∎

The proof of Theorem 3.3 given above is noneffective. To see if the proof could have been made effective we will look at a potential analogue. In order to state this analogue we need some definitions.

Definition 3.4 A tree T is *recursive* if the set T is recursive. A tree T is *recursively bounded* if T is recursive, T is bounded, and the bounding function is recursive.

Definition 3.5 A number $e = \langle e_1, e_2 \rangle$ *is an index for a recursively bounded tree* if (1) $e_1 \in TOT01$ and decides a set we denote T, (2) the set T is a tree, (3) $e_2 \in TOT$ and $\{e_2\}$ is a bounding function for T. T is the tree determined by e. The tree determined by e is denoted T_e. We denote the set of indices for recursively bounded trees by $TREE$. Note that $TREE$ is Π_2.

Potential Analogue 3.6 There is a recursive algorithm $\mathcal{A}$ that performs the following. Given an index e for an infinite recursively bounded tree T, $\mathcal{A}$ outputs an index for a recursive infinite branch. A consequence is that all infinite recursively bounded trees have recursive infinite branches.

3.2 Recursive Analogue is False

We show that the potential analogue is false. This was first shown by Kleene [104] and seems to be the first negative result in recursive combinatorics.

Definition 3.7 A pair of sets A and B is *recursively inseparable* if $A \cap B = \emptyset$ and there is no recursive R such that $A \subseteq R$ and $B \subseteq \overline{R}$.

Note 3.8 The sets $A = \{x : \varphi_x(x) \downarrow= 0\}$ and $B = \{x : \varphi_x(x) \downarrow= 1\}$ are easily seen to be a pair of r.e. recursively inseparable sets.

Theorem 3.9 *There exists an infinite recursively bounded tree with no infinite recursive branches.*

Proof: Let A, B be a pair of r.e. sets that are recursively inseparable. We define a recursively bounded tree T such that every infinite branch of T codes a set that separates A and B. Let T be defined by $\sigma \in T$ iff

1. $(\forall i < |\sigma|)\sigma(i) \in \{0, 1\}$,
2. $(\forall i < |\sigma|)i \in A_{|\sigma|} \Rightarrow \sigma(i) = 1$,
3. $(\forall i < |\sigma|)i \in B_{|\sigma|} \Rightarrow \sigma(i) = 0$.

Clearly T is recursively bounded and every infinite branch of T is the characteristic function of a set that separates A and B. Since A and B are recursively inseparable, T has no infinite recursive branches. ▮

Theorem 3.10 *The set*

$$RECBRANCH = \{e \mid T_e \text{ has a recursive infinite branch }\}.$$

is Σ_3-complete.

Proof:
$RECBRANCH$ consists of all $\langle e_1, e_2\rangle \in TREE$ such that there exists i with the following properties:

1. $i \in TOT01$.
2. $(\forall\sigma)[\{i\}(\sigma) = 1 \Rightarrow \{e_1\}(\sigma) = 1]$.
3. $(\forall\sigma)(\exists x)(\forall y \leq \{e_2\}(|\sigma| + 1), y \neq x)[(\{i\}(\sigma) = 1 \Rightarrow (\{i\}(\sigma x) = 1 \wedge \{i\}(\sigma y) = 0)]$.

Clearly $RECBRANCH$ is in Σ_3. To show that $RECBRANCH$ is Σ_3-complete we use the Σ_3-complete set

$$SEP = \{\langle x, y\rangle : W_x \text{ and } W_y \text{ are recursively separable}\}.$$

(For a proof that this set is Σ_3-complete see [159].) We show $SEP \leq_m RECBRANCH$.

Given $\langle x, y\rangle$ we define a recursively bounded tree T such that every infinite branch of T codes a set that separates W_x and W_y. Let T be defined by $\sigma \in T$ iff the following hold.

1. $(\forall i < |\sigma|)[\sigma(i) \in \{0, 1\}]$.
2. $(\forall i < |\sigma|)[i \in W_{x,|\sigma|} \Rightarrow \sigma(i) = 1]$.
3. $(\forall i < |\sigma|)[i \in W_{y,|\sigma|} \Rightarrow \sigma(i) = 0]$.

Clearly T is recursively bounded and every infinite branch of T is the characteristic function of a set that separates W_x and W_y. Hence T has an infinite recursive branch iff $\langle x, y\rangle \in SEP$. ∎

3.3 Recursion-Theoretic Modifications

Kreisel [109] showed that every infinite recursively bounded tree has a branch $B \leq_{\mathrm{T}} K$ (this follows easily from examining the proof of König's Lemma). Shoenfield [149] improved this to $B <_{\mathrm{T}} K$. Jockusch and Soare [90] improved this further to $B' \leq_{\mathrm{T}} K$. This is referred to as 'the low basis theorem.'

We introduce some notation which we will use later when applying the low basis theorem, then state the theorem in that notation.

Definition 3.11 A set of functions $\mathcal{F}$ is Π^0_1 if there exists a recursive predicate R such that $f \in \mathcal{F}$ iff $(\forall n)[R(\langle f(0), \ldots, f(n)\rangle)]$. If, in addition, there is a recursive function g such that $(\forall f \in \mathcal{F})(\forall n) f(n) \leq g(n)$, then $\mathcal{F}$ is called *a recursively bounded* Π^0_1 *class.* This definition easily relativizes to $\Pi^{0,A}_1$ class by taking $R \leq_{\mathrm{T}} A$.

Theorem 3.12 (Low Basis Theorem [90]) *If $\mathcal{F}$ is a nonempty recursively bounded Π^0_1 class, then there exists $f \in \mathcal{F}$ such that $f' \leq_T K$.*

There is also a relativized form.

Theorem 3.13 (Relativized Low Basis Theorem) *Let B be a set and $\mathcal{F}$ be a nonempty $\Pi_1^{0,B}$ class. If there exists $g \leq_T B$ such that $(\forall f \in \mathcal{F})(\forall n \in \mathsf{N})[f(n) \leq g(n)]$, then there exists $f \in \mathcal{F}$ such that $f' \leq_T B'$.*

To apply the theorem later, and to see that it yields a recursion-theoretic version of König's Lemma, we need the following proposition.

Proposition 3.14 *If T is a recursive (recursively bounded) tree, then the set of infinite branches forms a (recursively bounded) Π_1^0 class.*

Corollary 3.15 *If T is an infinite recursively bounded tree than T has an infinite branch B such that $B' \leq_{\mathrm{T}} K$.*

3.4 Miscellaneous

Carstens and Golze [29] considered adding some number of cross connections to the tree and seeing if it then had a recursive path. A *tree with cross connections* is a graph that can be looked at as a tree with some of the vertices on a level connected to each other. The n^{th} level is *saturated* if for all sets of n vertices on that level there exist two that are connected. A tree is *highly recursive* if, given a node, you can determine all of its children. Carstens and Golze showed that if G is a highly recursive tree with cross connections such that every level is saturated, then there exists a recursive infinite path. They were motivated by questions about one-dimensional cell spaces.

4 Ramsey's Theorem

We consider Ramsey's Theorem on colorings of $[\mathsf{N}]^2$. We (1) present the classical proof of the Ramsey's Theorem on colorings of $[\mathsf{N}]^2$, due to Ramsey [136, 137] (see [68] for several proofs), (2) show that a recursive analogue of Ramsey's Theorem is false and give an index-set version, (3) show that there are two recursion-theoretic modifications that are true, (4) state some results in proof theory that are related to this work, and (5) state some miscellaneous results.

4.1 Definitions and Classical Version

Definition 4.1 A *k-coloring of* $[\mathsf{N}]^m$ is a map from $[\mathsf{N}]^m$ into $\{1, 2, \ldots, k\}$. (It does not need to satisfy any additional properties.) The elements of $\{1, 2, \ldots, k\}$ are called *colors*. If $k = 2$, then we may refer to 1 as 'RED'and 2 as 'BLUE' (note that RED<BLUE).

Definition 4.2 Let c be a k-coloring of $[\mathsf{N}]^m$, i be a color, and $A \subseteq \mathsf{N}$. A is *i-homogeneous with respect to c* if (1) A is infinite, and (2) for all distinct $x_1, \ldots, x_m \in A$, $c(\{x_1, \ldots, x_m\}) = i$. A is *homogeneous with respect to c* if there is an i such that A is i-homogeneous with respect to c. We often drop the 'with respect to c' if the coloring is clear from context.

We exhibit a whimsical scenario to illustrate these concepts. Suppose that you host a party with a countably infinite number of guests. Assume their names are 0,1,2,…. Color each pair of guests RED if they know each other, and BLUE if they do not. This is a 2-coloring of $[\mathsf{N}]^2$. A RED-homogeneous set is an infinite set of people all of whom know each other, and a BLUE-homogeneous set is an infinite set of people no two of whom know each other.

We will later see that Ramsey's theorem (infinite version) guarantees that there is either a RED-homogeneous set or a BLUE-homogeneous set. This was first proven by Ramsey [136, 137]. For more on Ramsey theory (mostly finite versions) see [68].

We study the following simplified version of Ramsey's theorem (the general version involves m-colorings of $[\mathsf{N}]^k$). We give a direct proof; it can also be proven by König's Lemma (Theorem 3.3).

Theorem 4.3 *If c is a k-coloring of $[\mathsf{N}]^2$, then there exists a homogeneous set.*

Proof:

The variable d will range over the colors $\{1, \ldots, k\}$.

Let $A_0 = \mathsf{N}$ and $a_1 = 1$. Assume inductively that $A_{n-1} \subseteq \mathsf{N}$, $a_1, \ldots, a_n \in \mathsf{N}$, and $c_1, \ldots, c_{n-1} \in \{1, \ldots, k\}$ have been defined such that (1) A_{n-1} is infinite, (2) $a_1, \ldots, a_n$ are distinct, and (3) $(\forall i)(1 \le i \le n-1)(\forall x \in A_i - \{a_i\})$ $c(\{x, a_i\}) = c_i$. Let

$$\begin{aligned} c_n = \; & \mu d[|\{x : c(\{x, a_n\}) = d\} \cap A_{n-1}| = \infty] \text{ (exists since } A_{n-1} \text{ is infinite)}, \\ A_n = \; & \{x : c(\{x, a_n\}) = c_n\} \cap A_{n-1} \text{ (is infinite by the choice of } c_n), \\ a_{n+1} = \; & \mu x[x \in A_n - \{a_1, \ldots, a_n\}] \text{ (exists since } A_n \text{ is infinite)}. \end{aligned}$$

It is easy to see that these values satisfy (1),(2), and (3) above.

Let d be the least color that appears infinitely often in the sequence $c_1, c_2, \ldots$. Let $A = \{a_i : c_i = d\}$. It is easy to see that A is d-homogeneous. ∎

The proof of Theorem 4.3 given above is noneffective. To see if the proof could have been made effective, we look at the following potential analogue.

Potential Analogue 4.4 There is a recursive algorithm $\mathcal{A}$ that performs the following. Given an index e for a recursive 2-coloring of $[\mathsf{N}]^2$, $\mathcal{A}$ outputs an index for a recursive homogeneous set. A consequence is that all recursive 2-colorings of $[\mathsf{N}]^2$ induce a recursive homogeneous set.

Specker [161] showed that this Potential Analogue is false. We present a simpler proof by Jockusch [87]. We then show that some recursion-theoretic modifications are true and are the best possible. In particular, we will show the following results, all due to Jockusch.

1. There exists a recursive 2-coloring of $[\mathsf{N}]^2$ such that no homogeneous set is r.e.

2. There exists a recursive 2-coloring c of $[\mathsf{N}]^2$ such that no homogeneous set is recursive in K. This coloring c also induces no Σ_2 homogeneous sets.

3. Every recursive 2-coloring of $[\mathsf{N}]^2$ induces a Π_2 homogeneous set. This is the best possible in terms of the arithmetic hierarchy (by item *ii*).

4. Every recursive 2-coloring of $[\mathsf{N}]^2$ induces a homogeneous set A such that $A' \le_{\mathrm{T}} \emptyset''$.

4.2 Recursive Analogue is False

Definition 4.5 A set A is *bi-immune* if neither A nor $\overline{A}$ has an infinite r.e. subset.

Theorem 4.6 *There exists c, a recursive 2-coloring of $[\mathsf{N}]^2$, such that no homogeneous set is r.e.*

Proof:

Let X be a bi-immune set such that $X \leq_{\mathrm{T}} K$ (such is easily constructed by an initial-segment argument). By the limit lemma (see [159, p. 57]) there exists a 0-1 valued recursive function $f(x,s)$ such that $X(x) = \lim_{s\to\infty} f(x,s)$. Let c be the following coloring:

$$c(\{a,b\}) = \begin{cases} f(a,b)+1 & \text{if } a<b; \\ f(b,a)+1 & \text{if } b<a. \end{cases}$$

(The purpose of the '+1' is to ensure that range(c) $\subseteq \{1,2\}$.)

It is easy to see that a 1-homogeneous set is an infinite subset of $\overline{X}$, and a 2-homogeneous set is an infinite subset of X. Since X is bi-immune, there are no infinite r.e. subsets of X or $\overline{X}$, hence there are no r.e. infinite homogeneous sets. ∎

Theorem 4.7 *There exists c, a recursive 2-coloring of $[\mathsf{N}]^2$, such that no homogeneous set is recursive in K. This coloring also has no Σ_2 homogeneous set.*

Proof:

By the limit lemma, for every set $A \leq_{\mathrm{T}} K$ there exists a 0-1 valued primitive recursive f such that $A(x) = \lim_{s\to\infty} f(x,s)$. Let $f_0, f_1, f_2, \ldots$ be a standard enumeration of all 2-place 0-1 valued primitive recursive functions. Let A_i be the partial function defined by $A_i(x) = \lim_{s\to\infty} f_i(x,s)$ (if for x the limit does not exist, then $A_i(x)$ is not defined). Note that every set recursive in K is some A_i. We use A_i to represent both the set and its characteristic function.

We construct c, a recursive 2-coloring of $[\mathsf{N}]^2$, to satisfy the following requirements:

R_e: A_e total, infinite $\Rightarrow (\exists x, y, z \in A_e)[x, y, z$ distinct and $c(\{x, z\}) \neq c(\{y, z\})]$

It is easy to see that such a c will have no homogeneous set $A \leq_{\mathrm{T}} K$.

Let $A_e^s = \{x : f_e(x, s) = 1\} \cap \{0, 1, \ldots, s-1\}$.

CONSTRUCTION

Stage s: In this stage we will determine $c(\{s, 0\}), c(\{s, 1\}), \ldots, c(\{s, s-1\})$, and try to satisfy $R_0, \ldots, R_s$. For each R_e $(0 \leq e \leq s)$ in turn we look for the least two elements $x, y \in A_e^s$ such that $c(\{s, x\})$ and $c(\{s, y\})$ are not yet determined. If such an x and y exist, then set $c(\{s, x\}) = 1$ and $c(\{s, y\}) = 2$. After all the requirements are considered, set $c(\{s, x\}) = 1$ for all $x \leq s-1$ such that $c(\{s, x\})$ is not determined.

END OF CONSTRUCTION.

It is clear that c is recursive. We show that each R_e is satisfied. Assume A_e is total and infinite. Let $x_1 < x_2 < \cdots < x_{2e+2}$ be the first $2e+2$ elements of A_e. Let s' be such that for all $y \leq x_{2e+2}$, for all $t \geq s'$, $f_e(y, t) = A_e(y)$. Let s be the least stage such that $s \geq s'$ and $s \in A_e$. At stage s, when considering R_e, at most $2e$ of the pairs $\{s, x_i\}$ will have been colored (each $R_{e'}$, $0 \leq e' \leq e-1$, colors at most two pairs). Hence there exist $x, y \in \{x_1, \ldots, x_{2e+2}\} \subseteq A_e$ such that $c(\{s, x\}) \neq c(\{s, y\})$. Since $x, y, s \in A_e$, R_e is satisfied.

We now show that c has no homogeneous Σ_2 sets. Assume, by way of contradiction, that there is a Σ_2 homogeneous set. Every infinite Σ_2 set has an infinite subset that is recursive in K, hence there exists an infinite subset B (of the homogeneous set) such that $B \leq_{\mathrm{T}} K$. Since an infinite subset of a homogeneous set is homogeneous, B is a homogeneous set. Since $B \leq_{\mathrm{T}} K$, this contradicts the nature of c. ∎

4.3 How Hard is it to Tell if a Homogeneous set is Recursive?

In Section 4.2 we saw that it is possible for a recursive 2-coloring of $[\mathsf{N}]^2$ to not induce any recursive homogeneous sets. We examine how hard it is to tell if this is the case. We will show that the problem of determining if a coloring induces a recursive homogeneous set is Σ_3-complete.

Notation 4.8 Throughout this section let D be the set of indices for total recursive functions whose range is a subset of $\{1, 2\}$. We interpret elements of D as 2-colorings of $[\mathsf{N}]^2$.

Lemma 4.9 *There exists a recursive function f such that*
$x \in COF \Rightarrow \{f(x)\}^K$ *decides a finite set,*
$x \notin COF \Rightarrow \{f(x)\}^K$ *decides a bi-immune set.*

Proof:

Given x, we 'try' to construct a bi-immune set $A \leq_{\mathrm{T}} K$ by an initial-segment argument. If $x \in COF$, then our attempt will fail and A will be finite; however, if $x \notin COF$, then our attempt will succeed.

We try to construct A to satisfy the following requirements:
R_{2e} : W_e infinite $\Rightarrow (\exists y \in W_e - A)$,
R_{2e+1} : W_e infinite $\Rightarrow (\exists y \in W_e - \overline{A})$.

At the end of stage s of the construction, we will have (1) A_s, an initial segment of A, and (2) i_s, the index of the next requirement that needs to be satisfied. Note that if $|A_s| = n$, then A has been determined for $0, 1, \ldots, n-1$.

Formally the construction should be of an oracle Turing machine $\{f(x)\}^{()}$. Informally, we write it as a construction recursive in K, since the only oracle we use is K.

CONSTRUCTION of $\{f(x)\}^K$.

Stage 0: $A_0 = \lambda$ (the empty string). $i_0 = 0$.

Stage s+1: Ask K '$s \in W_x$?' If YES, then set $A_{s+1} = A_s 0$, $i_{s+1} = i_s$, and go to the next stage. If NO, then we work on satisfying R_{i_s}. There are two cases, depending on whether i_s is even or odd.

If $i_s = 2e$, then do the following: Ask K '$(\exists y \in W_e)[y \geq |A_s|]$?' (so $A(y)$ has not yet been determined). If NO, then W_e is finite, so R_{2e} is satisfied, hence we set $A_{s+1} = A_s 0$ and $i_{s+1} = i_s + 1$. If YES, then we set $A_{s+1} = A_s 0^{y+1-|A_s|}$ (so $A(y) = 0$) and $i_{s+1} = i_s + 1$.

If $i_s = 2e + 1$, then do the following. Ask K '$(\exists y \in W_e)[y \geq |A_s|]$?' (so $A(y)$ has not yet been determined). If NO, then W_e is finite, so R_{2e} is satisfied, hence we set $A_{s+1} = A_s 0$ and $i_{s+1} = i_s + 1$. If YES, then we set $A_{s+1} = A_s 0^{y-|A_s|} 1$ (so $A(y) = 1$) and $i_{s+1} = i_s + 1$.

END OF CONSTRUCTION

If $x \in COF$, then for almost all s we merely append 0 to A_s. Hence A is finite, and $\{f(x)\}^K$ decides a finite set.

If $x \notin COF$, then for every i, requirement R_i is satisfied at stage $s+1$, where s is the $i+1^{\text{st}}$ element of $\overline{W}_x$. Hence all requirements are satisfied, A is bi-immune, and $\{f(x)\}^K$ decides a bi-immune set.

▮

Theorem 4.10 *The set*

$$RECRAM = \{e : e \in TWOCOL \text{ and } \{e\} \text{ induces a recursive homog. set}\}$$

is Σ_3-complete.

Proof:

$RECRAM$ consists of all $e \in TWOCOL$ such that there exist i, d with the following properties:

1. $i \in TOT01$ and $d \in \{1, 2\}$,
2. $(\exists^\infty x)\{i\}(x) = 1$,
3. $(\forall x, y)[(\{i\}(x) = 1 \wedge \{i\}(y) = 1 \wedge x \neq y) \Rightarrow \{e\}(x, y) = d]$.

Clearly $RECRAM$ is Σ_3. To show that $RECRAM$ is Σ_3-hard we show that $COF \leq_m RECRAM$.

Given x, we produce an index for a recursive coloring of $[\mathsf{N}]^2$ such that $x \in COF$ iff that coloring has a recursive homogeneous set. Let f be defined as in Lemma 4.9. Let A_x be the set decided by $\{f(x)\}^K$ and let g be the total recursive 0-1 valued function such that $A_x(z) = \lim_{s \to \infty} g(z, s)$ (g exists by the limit lemma ([159, p. 57]). Let c be the coloring defined by

$$c(\{a, b\}) = \begin{cases} g(a, b) + 1 & \text{if } a < b; \\ g(b, a) + 1 & \text{if } b < a. \end{cases}$$

(The purpose of the '+1' is to ensure that range(c)$\subseteq \{1, 2\}$.) Let e be the index of this coloring. Note that e can be found effectively from x.

If $x \notin COF$, then A_x is bi-immune, so c is identical to the coloring in Theorem 4.6. Hence, by the reasoning used there, no homogeneous set can be Σ_1. Hence no homogeneous set can be recursive.

If $x \in COF$, then A_x is finite. We show that in this case there is a recursive homogeneous set, by defining a recursive increasing function h whose range is homogeneous. Let a be the least number such that none of the numbers $a, a+1, a+2, \ldots$ are in A_x.

$h(0) = a$

$h(n+1) = \mu b[\bigwedge_{i=0}^{n}(b > h(i) \wedge g(h(i), b) = 0)]$

By induction one can show that $h(n)$ is always defined. It is easy to see that the set of elements in the range of h forms a homogeneous set. ∎

4.4 Recursion-Theoretic Modifications

Theorem 4.11 *If c is a recursive k-coloring of $[\mathsf{N}]^2$ then there exists a Π_2 homogeneous set.*

Proof: We prove the $k = 2$ case. The general case is similar.

We essentially reprove Ramsey's Theorem carefully so that the homogeneous set is Π_2 (the usual proof yields a homogeneous set that is $\leq_{\mathrm{T}} \emptyset''$).

We construct a sequence of numbers $a_1 < a_2 < \cdots < a_i \cdots$ and a sequence of colors $c_1, c_2, \ldots, c_i, \ldots$ (we think of a_i as being colored c_i). The set R of red numbers will be Π_2 in any case, but possibly finite. If R is infinite, then it will be the homogeneous Π_2 set that we seek. If R is finite, then the set B of blue numbers will be Π_2 and infinite; hence it will be our desired Π_2 set.

We approximate the a_i's and c_i's in stages by a_i^s and c_i^s. We prove that both $\lim_{s\to\infty} a_i^s$ and $\lim_{s\to\infty} c_i^s$ exist.

By coloring a_i^s by c_i^s (for $1 \leq i \leq k$) we are guessing that there is an infinite number of n such that, for all $i \leq k$, $c(\{a_i^s, n\}) = c_i^s$. We will initially guess that a number is colored RED, but we may change our minds.

CONSTRUCTION

Stage 0: Set $a_1^0 = 0$ and color it RED, i.e. $c_1^0 =$RED.

Stage s+1: We have $a_1^s < \cdots < a_k^s$ (some k) and we want to extend it. Let M be larger than any number that has ever been an a_i^t for any $t \leq s$. Ask the following question (recursive in K): 'Does there exist $n > M$ such that, for all $i \leq k$, $c(\{a_i^s, n\}) = c_i^s$?'

If the answer is YES, then look for n until you find it. Set $a_{k+1}^{s+1} = n$ and $c_{k+1}^{s+1} =$RED, and for all $i \leq k$ set $a_i^{s+1} = a_i^s$ and $c_i^{s+1} = c_i^s$.

If the answer is NO, then by a series of similar questions find the value of $\max\{m : m \le k-1 \text{ and } (\exists n > M)(\forall i \le m)[c(\{a_i^s, n\}) = c_i^s]\}$ (the value $m = 0$ is permitted). Denote this number by m. Let $n > M$ be the least number such that $(\forall i \le m)[c(\{a_i^s, n\}) = c_i^s]$. Note that since m was maximum, the statement '$(\forall i \le m+1)[c(\{a_i^s, n\}) = c_i^s]$' is false. Hence $c(\{a_{m+1}^s, n\}) \neq c_{m+1}^s$. We will keep a_{m+1}^s but change its color, and discard all a_i^s with $i \ge m+2$. Formally, (1) set $a_{m+1}^{s+1} = a_{m+1}^s$, (2) set c_{m+1}^{s+1} to the opposite of what c_{m+1}^s was, (3) for $i \le m$ set $a_i^{s+1} = a_i^s$ and $c_i^{s+1} = c_i^s$, and (4) for $i \ge m+2$, a_i^{s+1} and c_i^{s+1} are undefined. The numbers a_i^s for $i \ge m+2$ are called *discarded*.

END OF CONSTRUCTION.

We show that (1) the sequences of a_i's and c_i's reach limits, and (2) either the set of RED numbers or the set of BLUE numbers is a Π_2 homogeneous set.

Claim 1: For all e, $\lim_{s\to\infty} a_e^s$ exists and $\lim_{s\to\infty} c_e^s$ exists.

Proof of Claim 1: Note that when a value of m is found in the NO case of the construction, then the elements discarded are those in places $m+2$ and larger. This observation will help prove the claim.

We proceed by induction on e. By the above observation, for $e = 1$, a_1^s is never discarded. If ever c_1^s turns BLUE, it is because for almost all x $c(\{a_1^s, x\})$ is BLUE. Hence it will never change color again.

Assume that the claim is true for $i < e$, hence for $1 \le i \le e-1$ there exists $a_i = \lim_{s\to\infty} a_i^s$ and $c_i = \lim_{s\to\infty} c_i^s$. Let s be the least number such that all the a_i^s and c_i^s $(1 \le i \le e-1)$ have settled down, i.e., for all $t \ge s$, $a_i^t = a_i$ and $c_i^t = c_i$. By the end of stage $s+1$, the values of a_e^{s+1} and c_e^{s+1} are defined. During stages $t \ge s$, if the NO case of the construction happens, then the resulting value of m will be $\ge e$. Hence a_e^t will never be discarded, so a_e^s has reached a limit, which we call a_e. Assume a_e changes to BLUE at some stage $t \ge s$. During stage t it was noted that for almost all n, if for all $i < e$, $c(\{a_i, n\}) = c_i$ then $c(\{a_e, n\}) =$BLUE. Hence a_e will never change color during a later stage. Hence $c_e = \lim_{s\to\infty} c_e^s$ exists.

End of Proof of Claim 1

Claim 2: Let d be either RED or BLUE. If $A = \{a_e : c_e = d\}$ is infinite, then it is homogeneous.

Proof of Claim 2: Let a_e, a_i be elements of A with $e < i$. Let s be the least number such that $a_e^s = a_e$ and $c_e^s = d$. Note that a_i must appear in the sequence at a stage past s, and that for all $t \geq s$, $a_e^s = a_e^t$ and $c_e^s = c_e^t$. By the construction, it must be the case that $c(\{a_e, a_i\}) = d$. Hence A is homogeneous.

End of Proof of Claim 2

Claim 3: Let $M = \{a_1, a_2, \ldots\}$, $R = \{a_i : c_i = \text{RED}\}$, $B = \{a_i : c_i = \text{BLUE}\}$. The set M is infinite. The sets M and R are Π_2. If R is finite, then B is infinite and Π_2.

Proof of Claim 3:

By Claim 1 and the construction M is infinite.

$x \in \overline{M}$ iff

$(\exists s)[[$ at stage s the YES case occurred with $n > x$ and $(\forall i) x \neq a_i^s] \vee$

$[x$ is discarded at stage $s]]$.

The matrix of this formula is recursive in K, so the entire formula can be rewritten in Σ_2 form. Hence M is Π_2.

$x \in \overline{R}$ iff

$x \notin M$ or $(\exists s)[x$ changes from RED to BLUE at stage $s]$.

(For x satisfying the second clause, x will either stay in M and be BLUE, or leave M. In either case, x is not RED.)

The second set is Σ_2, since the matrix of the formula is recursive in K. Hence $\overline{R}$ is the union of two Σ_2 sets, so it is Σ_2; therefore, R is Π_2.

If R is finite then $B = M - R$ is infinite. Note that the equation

$$x \in \overline{B} \text{ iff } x \notin M \text{ or } x \in R$$

yields a Σ_2 definition of $\overline{B}$, and hence B is Π_2.

End of proof of Claim 3

∎

Theorem 4.12 *If c is a recursive k-coloring of $[\mathsf{N}]^2$, then there exists a homogeneous set A such that $A' \leq_{\mathrm{T}} \emptyset''$.*

Proof: We prove the $k = 2$ case. The general case is similar.

We define a Π_1^0 class of functions $\mathcal{F}$ such that

1. $\mathcal{F}$ is nonempty,

2. there exists $g \leq_{\mathrm{T}} K$ such that $(\forall f \in \mathcal{F})(\forall n)[f(n) \leq g(n)]$, and

3. for every $f \in \mathcal{F}$ there exists $A \leq_{\mathrm{T}} f$ such that A is homogeneous.

The desired theorem will follow from Theorem 3.13. We describe $\mathcal{F}$ by describing a recursive tree T (see Proposition 3.14).

For $\sigma = (\langle a_0, c_0\rangle, \langle a_1, c_1\rangle, \ldots, \langle a_k, c_k\rangle)$, $\sigma \in T$ iff

1. $a_0 = 0$,
2. for all i, $0 \leq i \leq k$, $c_i \in \{1, 2\}$ (we think of the c_i as being colors), and
3. for all j, $1 \leq j \leq k$, $a_j = \mu x[x > a_{j-1} \wedge (\forall i \leq j-1) \;\; c(\{a_i, x\}) = c_i]$.

It is clear that testing $\sigma \in T$ is recursive. By a noneffective proof (similar to the construction of $a_0, a_1, \ldots$ in the proof of Theorem 4.3) one can show that T has an infinite branch, hence $\mathcal{F}$ is nonempty.

We now define a bounding function $g \leq_{\mathrm{T}} K$. We need two auxiliary functions, $NEXT$ and h.

Let $\sigma = (\langle a_0, c_0\rangle, \ldots, \langle a_n, c_n\rangle)$.

$$NEXT(\sigma) = \begin{cases} \mu x[x > a_n \wedge (\forall i \leq n) \;\; c(\{a_i, x\}) = c_i] & \text{if it exists and } \sigma \in T; \\ 0 & \text{otherwise.} \end{cases}$$

Note that $NEXT \leq_{\mathrm{T}} K$; and that if $\sigma = (\langle a_0, c_0\rangle, \ldots, \langle a_n, c_n\rangle)$ and $\sigma \in T$, then either

1. $NEXT(\sigma) = x \neq 0$, hence
   (a) $(\langle a_0, c_0\rangle, \ldots, \langle a_n, c_n\rangle, \langle x, 1\rangle) \in T$, and
   (b) $(\langle a_0, c_0\rangle, \ldots, \langle a_n, c_n\rangle, \langle x, 2\rangle) \in T$, or
2. $NEXT(\sigma) = 0$ and there are no extensions of σ on T.

Let h be defined by $h(0) = 1$, and

$$h(n+1) = \max_{a_1, \ldots, a_n \in \{1, 2, \ldots, h(n)\},\; c_0, \ldots, c_n \in \{1, 2\}} NEXT(\langle a_0, c_0\rangle, \ldots, \langle a_n, c_n\rangle).$$

Note that $h \leq_{\mathrm{T}} K$ and for all $\sigma \in T$, for all n, $h(n)$ bounds the first component of $\sigma(n)$.

Let $g(n) = \max_{x \leq h(n)}\{\langle x, 1\rangle, \langle x, 2\rangle\}$. Clearly $g \leq_{\mathrm{T}} K$ and for every $\sigma \in T$ and $n \in \mathsf{N}$, $\sigma(n) \leq g(n)$.

Let f be a function defined by an infinite branch of T (i.e., $f \in \mathcal{F}$). Let $c \in \{1, 2\}$ be such that $\exists^\infty n f(n) = \langle a_n, c\rangle$. Let $A = \{a : (\exists i) f(i) = \langle a, c\rangle\}$. The set A is homogeneous by the definition of f. Since f is increasing, $A \leq_{\mathrm{T}} f$. ∎

Theorem 4.12 has been improved by Cholak, Jockusch and Slaman [37]. We state the theorem but not the proof.

Theorem 4.13 *If c is a recursive k-coloring of $[\mathsf{N}]^2$, then there exists a homogeneous set A such that $A'' \leq_{\mathrm{T}} \emptyset''$.*

The proof uses the result by Jockusch and Stephan [91] that there is a low$_2$ r–cohesive set as well as a relativization of a new result of the authors that if $A_1, A_2, \ldots, A_n$ are Δ_2^0-sets and $\cup_{i=1}^n A_i = \mathsf{N}$, then some A_i has an infinite low$_2$ subset.

4.5 Connections to Proof Theory

We have only considered Ramsey's Theorem for coloring $[\mathsf{N}]^2$. The full theorem involves coloring $[\mathsf{N}]^m$.

Theorem 4.14 *Let $m \geq 1$. If c is a k-coloring of $[\mathsf{N}]^m$, then there exists a homogeneous set.*

Proof sketch: The proof is by induction on m and uses the technique of Theorem 4.3. ∎

The above theorem will henceforth be referred to as GRT (General Ramsey's Theorem). Jockusch [87] has shown the following 3 theorems. The proofs are similar to the proofs of Theorems 4.12, 4.11, and 4.7.

Theorem 4.15 *Let $m \geq 1$. If c is a recursive k-coloring of $[\mathsf{N}]^m$, then there exists a homogeneous set A such that $A' \leq_{\mathrm{T}} \emptyset^{(m)}$.*

Theorem 4.16 *Let $m \geq 1$. If c is a recursive k-coloring of $[\mathsf{N}]^m$, then there exists a homogeneous set $A \in \Pi_m$.*

Theorem 4.17 *Let $m \geq 2$. There exists a recursive 2-coloring c of $[\mathsf{N}]^m$ such that no homogeneous set is Σ_m.*

Several people observed that this last result has implications for proof theory [168]. We take a short digression into proof theory so that we can state the observation.

Proof theory deals with the strength required by an axiom system in order to prove a given theorem. Before discussing axioms, we need a language strong enough to state the theorem in question.

Definition 4.18 Let $\mathcal{L}_1$ be the language that contains the usual logical symbols, variables that we intend to range over N, constant symbols 0,1, and the usual arithmetic symbols $+, <, \times$. *Peano Arithmetic* (henceforth PA) is a set of axioms that establish the usual rules of addition and multiplication (e.g., associative) and allow a sentence to be proven by induction. (For details of the axioms see any elementary logic texts.)

PA suffices for most finite mathematics such as finite combinatorics and number theory. Gödel ([67], but see any elementary logic text for a proof in English) showed that there is a sentence ϕ that can be stated in $\mathcal{L}_1$ which is true but is not provable in PA. The sentence ϕ is not 'natural', in that it was explicitly designed to have this property. Using this result Gödel showed that the sentence 'PA is consistent' (which is expressible in $\mathcal{L}_1$) is not provable in PA. While this sentence is natural to a logician, it may not be natural to a non-logician. Later in this section we will see sentences that are natural (to a combinatorist) and true, but which are not provable in PA.

Note that GRT cannot be stated in $\mathcal{L}_1$. Hence we go to a richer language and axiom system.

Definition 4.19 Let $\mathcal{L}_2$ be $\mathcal{L}_1$ together with a second type of variable (for which we use capital letters) that is intended to range over subsets of N. Let CA (classical analysis) be the axioms of PA together with the following axioms:

1. $(\forall A)[(0 \in A \wedge (\forall x)(x \in A \Rightarrow x+1 \in A)) \Rightarrow (\forall x)(x \in A)]$ (induction).
2. For any formula $\phi(x)$ of $\mathcal{L}_2$ that does not contain the symbol 'A' we have the axiom $(\exists A)(\forall x)[x \in A \Leftrightarrow \phi(x)]$ (comprehension).

3. $(\forall A)(\forall B)[(\forall x)(x \in A \Leftrightarrow x \in B) \Rightarrow A = B]$ (extensionality).

The strength of CA can be moderated by restricting the comprehension axiom to certain types of formulas.

Definition 4.20 PPA is CA with the comprehension axiom weakened to use only arithmetic formulas (i.e., formulas with no bound set variables) for ϕ.

Notation 4.21 If S is an axiom system (e.g., PA) and ϕ is a sentence, then $S \vdash \phi$ means that there is a proof of ϕ from S, and $\neg(S \vdash \phi)$ means that there is no proof of ϕ from S.

We can now state and prove a theorem about how hard it is to prove GRT.

Theorem 4.22 *$\neg$(PPA$\vdash$ GRT).*

Proof sketch: Let $A(m, X, Y)$ be a formula in $\mathcal{L}_2$ with no bound set variables. If PPA$\vdash$ $(\forall m)(\forall X)(\exists Y)A(m, X, Y)$, then there exists an i such that PPA$\vdash$ $(\forall m)(\forall X)(\exists Y \in \Sigma_i^X)A(m, X, Y)$. (See [168] for proofs of a similar theorem by Jockusch and Solovay.)

We hope to make $(\forall m)(\forall X)(\exists Y)A(m, X, Y)$ the statement GRT. Let the notation $X \subseteq [\mathsf{N}]^m$ mean that X is a subset of N that we are interpreting via some fixed recursive bijection of N to $[\mathsf{N}]^m$ to be a subset of $[\mathsf{N}]^m$. This type of bijection can easily be described in PPA (actually in PA). Note that one can interpret $X \subseteq [\mathsf{N}]^m$ to mean that X represents a 2-coloring of $[\mathsf{N}]^m$ in which all elements of X are RED and all elements not in X are BLUE.

Let $A(m, X, Y)$ be the statement

$$X \subseteq [\mathsf{N}]^m \Rightarrow Y \text{ is homogeneous for the coloring } X.$$

Note that $(\forall m)(\forall X)(\exists Y)A(m, X, Y)$ is GRT. Assume, by way of contradiction, that PPA$\vdash$GRT. Then there exists i such that

PPA$\vdash$ $(\forall m)(\forall X)(\exists Y \in \Sigma_i^X)A(m, X, Y)$,

hence $(\forall m)(\forall X)(\exists Y \in \Sigma_i^X)A(m, X, Y)$ is true. If X is recursive then $\Sigma_i^X = \Sigma_i$. Hence we obtain

$$(*) \quad (\forall m)(\forall X, X \text{ recursive})(\exists Y \in \Sigma_i)A(m, X, Y).$$

Let $m > i$. By Theorem 4.17 there exists a recursive 2-coloring of $[\mathsf{N}]^m$ that has no Σ_m homogeneous sets. This coloring will have no Σ_i homogeneous sets.

$$(\exists m)(\exists X \text{recursive})(\forall Y \in \Sigma_i)[\neg A(m, X, Y)].$$

This contradicts $*$. Hence $\neg$(PPA$\vdash$GRT). ∎

Recall that Gödel showed there is a true sentence ϕ that is independent of a natural system (PA) but is unnatural (to a non-logician). By contrast, Theorem 4.22 exhibits a sentence (GRT) that is natural to some mathematicians (combinatorists) but is independent of an unnatural system (PPA). The question arises as to whether there are natural true sentences that are independent of natural systems. The answer is yes!

Paris and Harrington [133] proved the following variant of the finite Ramsey theorem to be independent of PA. Ketonen and Solovay [93] gave a different proof. A scaled down version of the proof is in Graham, Roth and Spencer's book on Ramsey theory [68].

Theorem 4.23 *Let $p, k, m \in \mathsf{N}$. There exists an $n = n(p, k, m)$ such that for all k-colorings of $[1, 2, \ldots, n]^m$ there exists a homogeneous set A such that $|A| \geq p$ and $|A|$ is larger than the least element of A. (Homogeneous is the usual definition except that the set is not infinite.)*

The independence proof of Paris and Harrington used model theory, while the proof of Ketonen and Solovay showed that the function $n(p, k, m)$ grows too fast to be proven to exist in PA. Both proofs are somewhat difficult. For an easier independence proof of a natural Ramsey-type theorem see Kanamori and McAloon [92]. For more on undecidability of Ramsey-type theorems see [20]. For other examples of natural theorems that are independent of natural systems see [155].

While it might be nice to say 'The independence results obtained by Jockusch were to foreshadow the later ones of Paris-Harrington and others', such a statement would be false. The observation that Jockusch's work on Ramsey theory leads to results on independence of true sentences from PPA happened after, and was inspired by, the Paris-Harrington results.

4.6 Miscellaneous

4.6.1 2-colorings of $[\mathsf{N}]^\omega$

There are versions of Ramsey's theorem that involve well-behaved colorings of $[\mathsf{N}]^\omega$, instead of arbitrary colorings of $[\mathsf{N}]^k$. Note that $[\mathsf{N}]^\omega$ is the set of all infinite subsets of N. View every element of $[\mathsf{N}]^\omega$ as an element of $\{0,1\}^\omega$, the characteristic string of the set represented by that element. View $[\mathsf{N}]^\omega$ as a topological space where X is a basic open set iff there exists $\sigma \in \{0,1\}^*$ such that $X = \{f \in [\mathsf{N}]^\omega \mid \sigma \prec f\}$. A coloring $c : [\mathsf{N}]^\omega \to \{1,\ldots,m\}$ is *clopen* (*Borel*, *analytic*) if there is an i such that $c^{-1}i)$ is clopen (Borel, analytic). Recall that clopen (in topology) means that both a set and its complement are the union of basic open sets.

We call a 2-coloring of $[\mathsf{N}]^\omega$ *Ramsey* if it induces a homogeneous set. Galvin and Prikry [60] showed that all Borel colorings are Ramsey. Silver [150] and Mathias [121] improved this by showing that all Σ_1^1 colorings are Ramsey. This result is essentially optimal, since there are models of set theory which have $\Sigma_2^1 \cap \Pi_2^1$ colorings that are not Ramsey (this happens when $V = L$); and there are models of set theory where all Π_2^1 colorings are Ramsey (this happens when there is a measurable cardinal [150]). An easier proof of the Silver-Mathias result was given by Ellentuck [50]. A game-theoretic proof that uses determinancy was given by Tanaka [163]. See [25] for a summary of this area.

A special case of the Galvin-Prikry theorem is that clopen colorings are Ramsey. A 2-coloring $c : [\mathsf{N}]^\omega \to \{1,2\}$ is *recursively clopen* if both $c^{-1}1)$ and $c^{-1}2)$ can be described as the set of extensions of some recursive subset of $[\mathsf{N}]^{<\omega}$. Simpson [151] showed that the recursive version of the clopen-Galvin-Prikry theorem is false in a strong way. He showed that for every recursive ordinal α there exists a recursively clopen 2-coloring such that if A is a homogeneous set then $\emptyset^\alpha \le_{\mathrm{T}} A$. Solovay [160] showed that every recursively clopen 2-coloring induces a hyperarithmetic homogeneous set. Clote [40] refined these results by looking at the order type of colorings.

4.6.2 Almost Homogeneous Sets

If c is a 2-coloring of $[\mathsf{N}]^2$, then an infinite set $A \subseteq \mathsf{N}$ is *almost homogeneous* if there exists a finite set F such that $A - F$ is homogeneous. A 2-coloring c of $[\mathsf{N}]^2$ is r.e. if either $c^{-1}RED)$ or $c^{-1}BLUE)$ is r.e. An infinite set A

is *m-cohesive* if for every r.e. coloring of $[\mathsf{N}]^m$, A is almost homogeneous (1-cohesive is the same as cohesive).

From the definition it is not obvious that there are sets that are m-cohesive. We show, in a purely combinatorial way (no recursion theory), that 2-cohesive sets exist (m-cohesive is similar). Let $c_1, c_2, \ldots$ be the list of all r.e. 2-colorings (for this proof they could be any countable class of 2-colorings that is closed under finite variations). Let $A_0 = \mathsf{N}$ and, for all $i > 0$, A_i is a subset of A_{i-1} which is homogeneous with respect to c_i. Let a_i be the least element of $A_i - \{a_0, \ldots, a_{i-1}\}$. The set $\{a_0, a_1, \ldots\}$ is 2-cohesive.

From the above proof it is not obvious that r.e. colorings have arithmetic 2-cohesive sets. Slaman [157] showed that there are Π_3 2-cohesive sets. Jockusch [88] showed that there are Π_2 2-cohesive sets. His proof combined the Friedberg-Yates technique used to construct a maximal set (see [159] Theorem X.3.3) with Jockusch's technique used to prove Theorem 4.11 (of this paper). This result is optimal in the sense that there is no Σ_2 2-cohesive set (by Theorem 4.7).

Hummel [85] is investigating, in her thesis, properties of k-cohesive sets. She has shown that there exists an r.e. 2-coloring c of $[\mathsf{N}]^2$ (i.e., one of $c^{-1}RED$) or $c^{-1}BLUE$) is r.e.) such that if B is a homogeneous set then $\emptyset' \le_{\mathrm{T}} B$; hence, if A is a 2-cohesive set then $\emptyset' \le_{\mathrm{T}} A$. She has also shown that there exists a recursive 2-coloring of $[\mathsf{N}]^2$ such that if B is a Π^0_2 homogeneous set then $\emptyset'' \le_{\mathrm{T}} B \oplus \emptyset'$; hence, if A is a 2-cohesive Π^0_2 set then $A \equiv_T \emptyset''$. The construction is the same as that in Theorem 4.7 used to obtain a 2-coloring such that if A is homogeneous then $A \not\le_{\mathrm{T}} K$. The proof that this construction yields this stronger result uses a result of Hummel which is analogous to Martin's result [120] that effectively simple sets are complete.

4.6.3 Degrees of Homogeneous Sets

By Theorem 4.7 there are 2-colorings of $[\mathsf{N}]^2$ that induce no recursive-in-K homogeneous sets. The question arises as to whether there are 2-colorings of $[\mathsf{N}]^2$ such that for all homogeneous sets A, $K \le_{\mathrm{T}} A$. There is such a 2-coloring of $[\mathsf{N}]^3$: color $\{a < b < c\}$ RED if $(\exists x < a)[x \in K_c - K_b]$, and BLUE otherwise (where $K_b(K_c)$ is the set containing the first b (c) elements of K in some fixed recursive enumeration). This result first appeared in [87] with a different proof. The proof here was communicated to me by Jockusch.

David Seetapun (reported in [85]) has shown that there is no such coloring of $[\mathsf{N}]^2$. Formally, he has shown that for any recursive 2-coloring of $[\mathsf{N}]^2$ there is a homogeneous set in which K is not recursive. The proof uses a forcing argument.

This problem arose because of possible implications in proof theory. In Paris's original paper on theorems unprovable in PA (see [132]) he has a version of Ramsey's theorem for 2-colorings of $[\mathsf{N}]^3$ that is not provable in PA. It uses ideas like the ones in the proof above about 2-coloring $[\mathsf{N}]^3$. If the above problem had been solved in the affirmative then there might be a version of Ramsey's theorem about 2-colorings of $[\mathsf{N}]^2$ that is not provable in PA. By contrast, Seetapun's result shows that Ramsey's theorem for $[\mathsf{N}]^2$ is not equivalent to arithmetic comprehension over RCA_0. (RCA_0 is a weak fragment of second order arithmetic where (roughly) only recursive sets can be proven to exist. RCA_0 stands for 'Recursive Comprehension Axiom'. See [152] for an exposition.)

Seetapun and Slaman [148] have shown that given any sequence of non-recursive sets $C_0, C_1, \ldots$ and any k-coloring of $[\mathsf{N}]^2$ there is an infinite homogenous set H such that $(\forall i)[C_i \not\leq_{\mathrm{T}} H]$. This result has implications for the proof-theoretic strenght of Ramsey's Theorem.

4.6.4 Dual Ramsey Theorem

There is a dual version of Ramsey's theorem that involves coloring the 2-colorings themselves. The Dual Ramsey Theorem is proven by Carlson and Simpson [24] and examined recursion-theoretically by Simpson [154].

4.6.5 Ramsey Theory and Peano Arithmetic

Clote [41] has investigated a version of Ramsey's theorem for coloring initial segments I of a model of Peano Arithmetic. He has constructed m-colorings of $[I]^n$ that have no Σ_n weakly-homogeneous subsets (this term is defined in his paper). From this he derives independence results for Peano Arithmetic.

5 Coloring Infinite Graphs

We consider vertex colorings of infinite graphs. We (1) present the theorem that a graph is k-colorable iff all of its finite subgraphs are k-colorable (originally due to de Bruijn and Erdos [45]), (2) show that a recursive analogue is false, (3) show a true combinatorial modification and show that it cannot be improved, (4) show that there is true recursion-theoretic modification, and (5) state some miscellaneous results.

5.1 Definitions and Classical Version

Definition 5.1 A graph $G = (V, E)$ is a set V (called *vertices*) together with a set E of unordered pairs of vertices (called *edges*). Edges of the form $\{v, v\}$ are not allowed. If $v \in V$ then the *degree of v in G* is $|\{x \in V : \{v, x\} \in E\}|$.

Definition 5.2 Let $G = (V, E)$ be a graph and $k \geq 1$. A *k-coloring of G* is a function c from V to $\{1, 2, \ldots, k\}$ such that no two adjacent vertices are assigned the same value. The values $\{1, 2, \ldots, k\}$ are commonly called *colors*. The *chromatic number of G*, denoted $\chi(G)$, is the minimal k such that G is k-colorable. If no such k exists, then by convention $\chi(G) = \infty$. If $G = (\emptyset, \emptyset)$, then by convention $\chi(G) = 0$. This is the only graph that is 0-colorable. A graph is *colorable* if there exists a $k \in \mathsf{N}$ such that $\chi(G) = k$.

We show that G is k-colorable iff every finite subgraph of G is k-colorable. We give a direct proof of the theorem; it can also be proven by König's Lemma (Theorem 3.3).

Theorem 5.3 *Let $G = (V, E)$ be a countable graph. G is k-colorable iff every finite subgraph of G is k-colorable.*

Proof: Assume every finite subgraph of G is k-colorable. Assume $V = \mathsf{N}$. Let $G_i = (V_i, E_i)$ be the finite graph defined by $V_i = \{0, 1, \ldots, i\}$ and $E_i = [\{0, 1. \ldots, i\}]^2 \cap E$. Since G_i is a finite subgraph of G, G_i is k-colorable. Let c_i be a k-coloring of G_i that uses $\{1, 2, \ldots, k\}$ for its colors. We use the c_i to define a k-coloring c of G. Let

$$\begin{aligned} c(0) = &\ \mu x[(\exists^\infty i) c_i(0) = x] \\ c(n+1) = &\ \mu x[(\exists^\infty i)((\textstyle\bigwedge_{y=0}^{n} c_i(y) = c(y)) \wedge (c_i(n+1) = x))] \end{aligned}$$

It is easy to see that c is a k-coloring of G.

The converse is obvious. ∎

The proof of Theorem 5.3 given above is noneffective. To see if the proof could have been made effective we will look at a potential analogue. In order to state this analogue we need some definitions.

Definition 5.4 A graph $G = (V, E)$ is *recursive* if $V \subseteq \mathsf{N}$ and $E \subseteq [\mathsf{N}]^2$ are recursive.

Definition 5.5 A graph $G = (V, E)$ is *highly recursive* if every vertex of G has finite degree and the function that maps a vertex to an encoding of the set of its neighbors is recursive.

Definition 5.6 Let $G = (V, E)$ be a graph such that $V \subseteq \mathsf{N}$, and let $k \geq 1$ (in practice G will be a recursive or highly recursive graph). G is *recursively k-colorable* if there exists a recursive function c that is a k-coloring of G. The *recursive chromatic number of* G, denoted by $\chi^r(G)$, is the minimal k such that G is recursively k-colorable. If no such k exists, then by convention $\chi^r(G) = \infty$. If $G = (\emptyset, \emptyset)$, then by convention $\chi^r(G) = 0$. This is the only graph that is recursively 0-colorable. A graph is *recursively colorable* if there exists a $k \in \mathsf{N}$ such that $\chi^r(G) = k$.

We will represent recursive graphs by the Turing machines that determine their vertex and edge sets. An index for a recursive graph will be an ordered pair, the first component of which is an index for a Turing machine which decides the vertex set, the second the edge set.

Definition 5.7 A number $e = \langle e_1, e_2 \rangle$ *determines a recursive graph* if $e_1, e_2 \in TOT01$. The *graph that* $\langle e_1, e_2 \rangle$ *determines*, denoted by G^r_e, has vertex set $V = \{x : \{e_1\}(x) = 1\}$ and edge set $E = \{\{x, y\} : x, y \in V, x \neq y, \{e_2\}(x, y) = \{e_2\}(y, x) = 1\}$.

Definition 5.8 A number $\langle e_1, e_2 \rangle$ *determines a highly recursive graph* if $e_1 \in TOT01$, $e_2 \in TOT$, and if $\{e_2\}$ is interpreted as mapping numbers to finite sets of numbers then $x \in \{e_2\}(y)$ iff $y \in \{e_2\}(x)$. The *graph that* $\langle e_1, e_2 \rangle$ *determines*, denoted by G^{hr}_e, has vertex set $V = \{x : \{e_1\}(x) = 1\}$ and edge set $E = \{\{x, y\} : x, y \in V, x \neq y, x \in \{e_2\}(y)\}$.

Potential Analogue 5.9 There is a recursive algorithm $\mathcal{A}$ that performs the following. Given (1) an index e for recursive graph G_e^r, and (2) an index i for a recursive function that will k-color any finite subgraph of G_e^r, $\mathcal{A}$ outputs an index for a recursive k-coloring of G_e^r. A consequence is that every recursive k-colorable graph would be recursively k-colorable. (A similar analogue and consequence can be stated for highly recursive graphs.)

We will soon (Theorem 5.15) see that this potential analogue is false. Hence we will look at modifications of it. There are two parameters to relax: either we can settle for a recursive $f(k)$-coloring (f is some function) instead of a k-coloring, or we can settle for a k-coloring that is not that strong in terms of Turing degree (the coloring will turn out to be of low Turing degree).

We will show the following.

1. There exists a recursive graph G such that $\chi(G) = 2$ but $\chi^r(G) = \infty$. Hence Potential Analogue 5.9 is false. Moreover, no modification allowing more colors is true.

2. There exists a highly recursive graph G such that $\chi(G) = k$ but $\chi^r(G) = 2k - 1$. Hence Potential Analogue 5.9 is false even for highly recursive graphs.

3. If G is a highly recursive graph that is k-colorable, then one can effectively produce (given an index for G) a recursive $(2k-1)$-coloring of G. Hence a combinatorial modification of Potential Analogue 5.9 is true.

4. If G is a recursive graph that is k-colorable, there is a k-coloring of low degree. Hence a recursion-theoretic modification of Potential Analogue 5.9 is true.

We will need the following definitions from graph theory.

Definition 5.10 Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs such that $V_1, V_2 \subseteq \mathsf{N}$ (in practice they will be recursive graphs). G_1 and G_2 are *isomorphic* if there is a map from V_1 to V_2 that preserves edges. We denote this by $G_1 \cong G_2$.

Notation 5.11 Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs. $G_1 \subseteq G_2$ means that $V_1 \subseteq V_2$ and $E_1 \subseteq E_2$. $G = G_1 \cup G_2$ is the graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$.

5.2 Recursive Analogue is False for Recursive Graphs

We show that for all a, b such that $2 \leq a < b \leq \infty$, there exists a recursive graph G such that $\chi(G) = a$ and $\chi^r(G) = b$. The lemmas we prove are more general than we need for this application, however they will be used again in Section 5.3.

Definition 5.12 If $\{e\}$ is a Turing machine and W is a set such that, for all $x \in W$, $\{e\}(x)\downarrow$, then $\{e\}(W) = \{\{e\}(x) : x \in W\}$.

The next lemma is implicit in [10], though it was stated and proven in [13].

Lemma 5.13 *Let $i \geq 0$, $\{e\}$ be a Turing machine, and X be an infinite recursive set. There exists a finite sequence of finite graphs $L_1, \ldots, L_r$ such that the following conditions hold. (For notation $L_j = (V_j, E_j)$.)*

1. *L_1 is a graph consisting of 2^i isolated vertices. For every j, $2 \leq j \leq r$, $V_{j-1} \subseteq V_j$ and $E_{j-1} \subseteq E_j$. For each j, $1 \leq j \leq r$, (1) $V_j \subseteq X$, and (2) canonical indices for the finite sets V_j and E_j can be effectively computed given e, i, j and an index for X.*

2. *For every j, $1 \leq j < r$, (1) for all $x \in V_j$, $\{e\}(x)\downarrow$, and (2) L_{j+1} can be obtained recursively from L_j and the values of $\{e\}(x)$ for every $x \in V_j$.*

3. *There exists a nonempty set $W \subseteq V_r$ of vertices such that either*
   - *(a) $\{e\}$ is not total on W (so $\{e\}$ is not a coloring of L_r), or*
   - *(b) there exist $v \in V_r$, $w \in W$ such that $\{v, w\} \in E_r$ and $\{e\}(v) = \{e\}(w)$ (so $\{e\}$ is not a coloring of L_r), or*
   - *(c) for all $x \in W$, $\{e\}(x)\downarrow$, and $|\{e\}(W)| = i + 1$ (so $\{e\}$ is not an i-coloring of L_r).*

4. *There is a 2-coloring of L_r in which W is 1-colored. Hence $\chi(L_r) \leq 2$.*

5. *An index for a recursive $(i+1)$-coloring of L_j can be effectively obtained from e, i, j and an index for X.*

6. *L_r is planar.*

The set W witnesses the fact that $\{e\}$ is not an i-coloring of L_r. We call W a witness of type 1, 2, *or* 3, *depending on which subcase of* (c) *it falls under. If it falls under more than one, then we take the least such subcase.*

Proof:

The Turing machine $\{e\}$ is fixed throughout this proof.

We prove this lemma by induction on i. Assume $i = 0$ and x is the first element of X. Let $L_1 = L_r = (\{x\}, \emptyset)$ and $W = \{x\}$. If $\{e\}(x) \uparrow$, then W is a witness of type 1. If $\{e\}(x) \downarrow$, then W is a witness of type 3. In either case conditions i–vi are easily seen to be satisfied.

Assume this lemma is true for i. We show it is true for $i+1$. Let $X = Y \cup Z$ be a recursive partition of X into infinite recursive sets such that indices for Y and Z can be obtained from indices for X. Apply the induction hypothesis to the values i, e, Y and also to i, e, Z to obtain the following:

1. a sequence of graphs $L_{11}, L_{21}, \cdots, L_{r_1 1}$, and a set W_1 such that the sequence together with witness set W_1 satisfies i–vi (note that all the vertices are in Y), and

2. a sequence of graphs $L_{12}, L_{22} \cdots, L_{r_2 2}$, and a set W_2 such that the sequence together with witness set W_2 satisfies i–vi (note that all the vertices are in Z).

Assume $r_1 \le r_2$. We define graphs $L_1, L_2, \ldots, L_{r'}$ that satisfy the theorem (r' will be either r_1, r_2 or $r_2 + 1$). For $1 \le j \le r_1$ let

$$L_j = L_{j1} \cup L_{j2}.$$

If for all x, x a vertex of $L_{r_1 1}$, $\{e\}(x) \downarrow$, then for $r_1 + 1 \le j \le r_2$ let

$$L_j = L_{r_1 1} \cup L_{j2}.$$

(If this does not occur, then L_{r_1} is the final graph and W_1 is the witness set.) In this case we obtain witness sets as follows. If W_1 (W_2) is a witness of type 1 or 2, then L_{r_2} is our final graph and $W = W_1$ (W_2). The 2-coloring of the final graph with the witnesses 1-colored can be obtained by combining such colorings from $L_{r_1 1}$ and $L_{r_2 2}$. It is easy to see that the sequence of graphs and the witness set W all satisfy requirements i–vi.

If both W_1 and W_2 are witnesses of type 3, then there are two cases:

(Case 1) If $\{e\}(W_1) \neq \{e\}(W_2)$, then either there is some element $w \in W_1$ such that $\{e\}(w) \notin \{e\}(W_2)$, or there is some element $w \in W_2$ such that $\{e\}(w) \notin \{e\}(W_1)$. We examine the latter case, the former is similar. Our final graph is L_{r_2} and we let $W = W_1 \cup \{w\}$. By the induction hypothesis and the fact that W_1 is of type 3, $|\{e\}(W_1)| = i + 1$. Since $w \notin W_1$ and $\{e\}(w) \notin \{e\}(W_1)$, $|\{e\}(W_1 \cup \{w\})| = i + 2$. Hence W is a witness of type 3. The 2-coloring of the final graph with the witnesses 1-colored can be obtained by combining such colorings from $L_{r_1 1}$ and $L_{r_2 2}$.

(Case 2) If $\{e\}(W_1) = \{e\}(W_2)$, then let w be the least element of X that is bigger than both any element used so far and the number of steps spent on this construction so far (this is done to make the graph recursive). Let

$$\begin{aligned} L_{r_2+1} = & \ L_{r_2} \cup \{\{u, w\} : u \in W_1\} \\ W = & \ W_2 \cup \{w\}. \end{aligned}$$

If $\{e\}(w) \uparrow$, then W is a witness of type 1. If $\{e\}(w) \downarrow \in \{e\}(W_1)$, then since w is connected to all vertices in W_1, W is a witness of type 2. If $\{e\}(w) \downarrow \notin \{e\}(W_1)$ (and hence $\{e\}(w) \notin \{e\}(W_2)$) then $\{e\}(W) = \{e\}(W_2 \cup \{w\}) = \{e\}(W_2) \cup \{e\}(w)$, which has cardinality $i + 2$; hence W is a witness of type 3. Hence W is a witness set. A 2-coloring of L_{r_2+1} with W 1-colored can easily be obtained from the 2-coloring of $L_{r_1 1}$ (that 1-colors W_1) and the 2-coloring of $L_{r_2 2}$ (that 1-colors W_2).

It is easy to see that the sequence $L_1, L_2, \cdots, L_{r'}$ and the set W satisfy i,ii,iii,iv,v,vi. Given e, i, j and an index for X one can effectively find indices for Y, Z and then use the induction hypothesis and the construction to obtain an index for an $(i + 1)$-coloring of L_j; hence v holds. ∎

Lemma 5.14 *Let $a \geq 2, i \geq a$, $\{e\}$ be a Turing machine, and X be an infinite recursive set. There exists a finite sequence of finite graphs $L_1, \ldots, L_r$ such that the following conditions hold. (For notation $L_j = (V_j, E_j)$.)*

1. *Every L_j is the union of an a-clique $(A, [A]^2)$ and a 2-colorable graph. For every j, $2 \leq j \leq r$, $V_{j-1} \subseteq V_j$ and $E_{j-1} \subseteq E_j$. For every j, $1 \leq j \leq r$, (1) $V_j \subseteq X$, and (2) canonical indices for the finite sets V_j and E_j can be effectively computed given e, i, j and an index for X.*

2. *For every j, $1 \leq j < r$, L_{j+1} can be obtained recursively from L_j and the values of $\{e\}(x)$ for every $x \in V_j - A$.*

3. *$\chi(L_r) = a$ (this is the difference between this lemma and Lemma 5.13).*

4. *An index for a recursive $(i+1)$-coloring of L_j can be effectively obtained from a, e, i, j and an index for X.*

5. *Every L_j is the union of an a-clique and a planar graph.*

Proof: Let $x_1, \ldots, x_a$ be the first a elements of X. Let K_a be the a-clique on the vertices $\{x_1, \ldots, x_a\}$. Let $L'_1, \ldots, L'_r$ be the sequence obtained from applying Lemma 5.13 to $e, i, X - \{x_1, \ldots, x_a\}$. For all j, $1 \leq j \leq r$, let $L_j = L'_j \cup K_a$. ∎

Theorem 5.15 *Let a, b be such that $2 \leq a < b \leq \infty$. Let X be an infinite recursive set. There exists a recursive graph $G = (V, E)$ such that $\chi(G) = a$, $\chi^r(G) = b$, and $V \subseteq X$. If $a \leq 4$ then G can be taken to be planar.*

Proof:
Recursively partition X into sets $U_{\langle e,i \rangle}$ such that $U_{\langle e,i \rangle}$ is infinite. Let $G(e, i)$ be the graph constructed in Lemma 5.14 using parameters $a, e, i, U_{\langle e,i \rangle}$ (i.e., $G(e, i)$ is the graph called 'L_r'). Let $G = \bigcup_{e=0}^{\infty} \bigcup_{0 \leq i < b} G(e, i)$. Clearly G is recursive and $\chi(G) = a$. Since $(\forall e)(\forall i < b)$, $\{e\}$ is not an i-coloring of $G(e, i)$, we have $\chi^r(G) \geq b$. Since $(\forall e)(\forall i < b)$, $\chi^r(G(e, i)) \leq i + 1$ in a uniform way, $\chi^r(G) \leq b$. Combining these inequalities yields $\chi^r(G) = b$. ∎

Corollary 5.16 *There exists a recursive graph G such that $\chi(G) = 2$, $\chi^r(G) = \infty$, and G is planar.*

Note 5.17 The corollary (and its proof) are essentially due to Bean, who actually showed that there exists a *connected* recursive graph G such that $\chi(G) = 3$ and $\chi^r(G) = \infty$. Both Bean's result and ours are optimal since

1. for connected recursive graphs G, $\chi(G) = 2 \Rightarrow \chi^r(G) = 2$, and

2. for recursive graphs G, $\chi(G) \leq 1 \Rightarrow \chi^r(G) = \chi(G)$.

In addition, both the graph in Bean's proof and the graph in Corollary 5.16 are planar.

5.3 How Hard is it to Determine $\chi^r(G)$?

Theorem 5.15 says that there are recursive graphs G such that $\chi(G)$ and $\chi^r(G)$ are very different. Given a graph, how hard is it to tell if it is of this type? In this section we show that, even if $\chi(G)$ is known and $\chi^r(G)$ is narrowed down to two values, it is Σ_3-complete to determine $\chi^r(G)$. By contrast the following promise problem is Π_1-complete: (D, A), where

$$D = \{e \mid \ e \text{ is the index of a recursive graph}\}$$

and

$$A = \{e \in D \mid \ \text{the graph represented by } e \text{ is } k\text{-colorable}\}.$$

Lemma 5.18 *Let $a \geq 2, i \geq a$, $\{e\}$ a Turing machine, and X an infinite recursive set. There exists an infinite sequence of (not necessarily distinct) finite graphs $H_1, H_2, \ldots$ such that the following hold. (For notation $H_s = (V_s, E_s)$.)*

1. $V_1 \subseteq V_2 \cdots$.
2. *For all s, $V_s \subseteq X$ and $\chi(H_s) = a$.*
3. *Given a, e, i, s and an index for X one can effectively find canonical indices for the finite sets V_s and E_s.*
4. *There exists a finite graph H and a number t such that $(\forall s \geq t) H_s = H$. We call this graph $\lim_{s\to\infty} H_s$.*
5. *H is not i-colored by $\{e\}$.*
6. $\chi(H) = a$.
7. *Given a, e, i, s and an index for X one can effectively find an index for a recursive $(i+1)$-coloring of H_s.*

Proof:

Apply Lemma 5.14 to the parameters a, e, i, X. View the construction of L_r as proceeding in stages where, at each stage, only one more step in the construction is executed. Let H_s be the graph produced at the end of stage s. It is easy to see that i–vii are satisfied. ∎

Lemma 5.19 *Let $a \geq 2, i \geq a$, $\{e\}$ be a Turing machine, and X be an infinite recursive set. Let $y \in \mathsf{N}$. There exists a recursive graph $G = (V, E)$, which depends on y, such that the following hold.*

1. *$V \subseteq X$.*
2. *Given a, e, i, y and an index for X, one can effectively find an index for G.*
3. *Every component of G is finite.*
4. *$\chi(G) = a$.*
5. *If $y \notin TOT$ then*
   - *(a) G consists of a finite number of finite components, and*
   - *(b) G is not i-colored by $\{e\}$.*
6. *If $y \in TOT$ then*
   - *(a) G consists of an infinite number of finite components,*
   - *(b) given a, e, i, y and an index for X, and $v \in V$, one can effectively find the finite component containing v, and*
   - *(c) given a, e, i, y and an index for X one can find an index for a recursive a-coloring of G (this follows from $\chi(G) = a$ and items vi.a and vi.b).*

Proof:

We consider a, e, i, y and X fixed throughout this proof. Let $X = \bigcup_{j=0}^{\infty} X_j$ be a recursive partition of X into an infinite number of infinite recursive sets. Let $H_1(j), H_2(j), \ldots$ be the sequence of graphs obtained by applying Lemma 5.18 to parameters a, e, i, X_j. We use these graphs to construct G in stages.

CONSTRUCTION

Stage 0: $G_0 = (\emptyset, \emptyset)$.

Stage s+1: Let j_s be the least element that is not in $W_{y,s}$. Let G_{s+1} be $G_s \cup H_s(j_s)$. Note that if $j_s \neq j_{s-1}$ then a new component is started.

END OF CONSTRUCTION

Let $G = \bigcup_s G_s$. It is clear that G satisfies i and ii. Since for every j both $H_s(j)$ and $H = \lim_{s\to\infty} H_s(j)$ are finite, iii holds. By Lemma 5.18 each component of G is a-colorable, therefore G is a-colorable. Hence iv holds.

Assume $y \notin TOT$. Let j be the least element of $\overline{W}_y$. Let t be the least stage such that $0, 1, \ldots, j-1 \in W_{y,t}$. For all $s \geq t$, $j_s = j$; therefore G consists of a finite number of graphs of the form $H_{s'}(j')$ (where $s' < t$ and $j' < j$) along with $H = \lim_{s\to\infty} H_s(j)$. Hence $v.a$ holds. By Lemma 5.18, H is not i-colored by $\{e\}$, hence $v.b$ holds.

Assume $y \in TOT$. Since $W_y = \mathsf{N}$, $\lim_{s\to\infty} j_s = \infty$. During every stage s such that $j_s \neq j_{s+1}$ a new component is created; therefore G consists of an infinite number of components. Hence $vi.a$ holds.

To establish $vi.b$ we show, given $v \in V$, how to find all the vertices and edges in the finite component containing v. Run the construction until $j, s \in \mathsf{N}$ are found such that v is a vertex of $H_s(j)$ (this will happen since $v \in V$). Run the construction further until t is found such that $j < j_t$ (this will happen since $y \in TOT$). The finite component of $H_t(j)$ that contains v is the finite component of G that contains v. ∎

Theorem 5.20 *Let $a, b \in \{2, 3, \ldots\} \cup \{\infty\}$ where $a < b$. Let D be the set of indices of recursive graphs with chromatic number a and recursive chromatic number either a or b. Let $RECCOL_{a,b}$ be the 0-1 valued partial function defined by*

$$RECCOL_{a,b}(e) = \begin{cases} 1 & \text{if } e \in D \text{ and } \chi^r(G_e^r) = a; \\ 0 & \text{if } e \in D \text{ and } \chi^r(G_e^r) = b; \\ \text{undefined} & \text{if } e \notin D \ . \end{cases}$$

The promise problem $(D, RECCOL_{a,b})$ is Σ_3-complete.

Proof:

Let TOT_a be the set of indices for total Turing machines whose image is contained in $\{1, \ldots, a\}$. Note that TOT_a is Π_2. The set A defined below is a Σ_3 solution of $(D, RECCOL_{a,b})$.

A is the set of ordered pairs $\langle e_1, e_2 \rangle$ such that $e_1 \in TOT01 \wedge e_2 \in TOT01$ and there exists i such that

1. $i \in TOT_a$, and

2. $(\forall x, y)[(x \neq y \wedge \{e_1\}(x) = \{e_1\}(y) = 1 \wedge \{i\}(x) = \{i\}(y)) \Rightarrow \{e_2\}(x, y) = 0]$

This definition of A can easily be put into Σ_3 form.

We show that $(D, RECCOL_{a,b})$ is Σ_3-hard by showing that if A is a solution to $(D, RECCOL_{a,b})$ then $COF \leq_{\mathrm{m}} A$. Given x, we construct a recursive graph $G(x) = G$ such that

$x \in COF \Rightarrow \chi^r(G) = a$, and

$x \notin COF \Rightarrow \chi^r(G) = b$.

We use a modification of the construction in Theorem 5.15 of a recursive graph G such that $\chi(G) = a$ but $\chi^r(G) = b$. In this modification we weave the set W_x into the construction in such a way that if W_x is cofinite, then the construction fails and $\chi^r(G) = a$; and if W_x is not cofinite then the construction succeeds and $\chi^r(G) = b$.

Let $\mathsf{N} = \bigcup_{e,i} X_e^i$ be a recursive partition of N into an infinite number of infinite recursive sets. Let $y_{\langle e,i\rangle}$ be defined such that

$$y_{\langle e,i\rangle} \in TOT \text{ iff } \{\langle e,i\rangle, \langle e,i\rangle + 1, \ldots\} \subseteq W_x.$$

Let $G(e,i) = (V(e,i), E(e,i))$ be the recursive graph obtained by applying Lemma 5.19 to $a, e, i, X_e^i, y_{\langle e,i\rangle}$. Let $G = \bigcup_e \bigcup_{0 \leq i < b} G(e,i)$ and $G = (V, E)$. Clearly G is recursive and $\chi(G) = a$.

If $x \notin COF$ then for all e, i we have $y_{\langle e,i\rangle} \notin TOT$. Hence, by Lemma 5.19, for all e and all $i < b$, $G(e,i)$ is not i-colored by $\{e\}$. Therefore $\chi^r(G) \geq b$. By Lemma 5.19 (item vi.c), the graphs $G(e,i)$ are recursively $(i+1)$-colorable in a uniform way, hence $\chi^r(G) \leq b$. Combining these two yields $\chi^r(G) = b$. (Note that this argument holds when $b = \infty$.)

If $x \in COF$ then $S' = \{\langle e,i\rangle \mid y_{\langle e,i\rangle} \notin TOT \wedge 0 \leq i < b \wedge e \in \mathsf{N}\}$ is finite. Let $S'' = \{\langle e,i\rangle \mid y_{\langle e,i\rangle} \in TOT \wedge 0 \leq i < b \wedge e \in \mathsf{N}\}$, $G' = \bigcup_{\langle e,i\rangle \in S'} G(e,i)$ and $G'' = \bigcup_{\langle e,i\rangle \in S''} G(e,i)$. Note that $G = G' \cup G''$. We show that $G' \cup G''$ is recursively a-colorable by showing that G' and G'' are recursively a-colorable (and using that $G = G' \cup G''$ is a recursive partition of G).

If $\langle e,i\rangle \in S'$ then $y_{\langle e,i\rangle} \notin TOT$ so, by Lemma 5.19, $G(e,i)$ is finite and $\chi(G(e,i)) = a$. Since S' is finite, G' is a finite a-colorable graph. Hence $\chi^r(G') = a$.

If $\langle e,i\rangle \in S''$ then $y_{\langle e,i\rangle} \in TOT$ so, by Lemma 5.19, one can effectively find the finite component of $G(e,i)$ in which a given $v \in V(e,i)$ is contained. We use this to recursively a-color G''. Let $G'' = (V'', E'')$.

Given a number v, first check if it is in V''. If it is not then output 1 and halt (we need not color it). If $v \in V''$ then run the construction until you find e, i such that $v \in G(e,i)$. Then find the finite component of $G(e,i)$ that contains v. Let c be the least lexicographic coloring of this component.

$\blacksquare$

5.4 Combinatorial Modification

Theorem 5.15 shows that there are recursive graphs G such that, even though $\chi(G) = 2$, no finite number of colors will suffice to color it recursively. If G is highly recursive, then more colors do help. The following theorem was first proven in [145] but also appears in [30] (independently).

Theorem 5.21 *If G is highly recursive and n-colorable, then G is recursively $(2n-1)$-colorable. Moreover, given an index for G (as a highly recursive graph), one can recursively find an index for a $(2n-1)$-coloring of G.*

Proof: Assume, without loss of generality, that $V = \mathsf{N}$. Color vertex 1 with color 1. Assume the following inductively

1. The vertices $\{1, \ldots, m\}$ are colored with $\{1, \ldots, 2n-1\}$ (additional vertices may also be colored with $\{1, \ldots, 2n-1\}$, but not with any other colors).

2. Let $B_m = \{v \mid v$ is colored and v is adjacent to a vertex that is not colored$\}$. The vertices of B_m are colored with either $\{1, \ldots, n-1\}$ or $\{n+1, \ldots, 2n-1\}$.

We color the vertex $m+1$ (and possibly some additional vertices). If $m+1$ is already colored, then note that (1) and (2) hold for $m+1$, so proceed to color $m+2$. Otherwise we color $m+1$ as follows. Let H be the set

$$\{v \mid v \text{ is not colored}, \exists u \text{ that is colored such that } d(u,v) \leq 2\} \cup \{m+1\}.$$

($d(u,v)$ is the length of the shortest path from u to v.) Assume B_m is colored with $\{1, \ldots, n-1\}$ (the case where B_m is colored with $\{n+1, \ldots, 2n-1\}$ is similar). Color H with $\{n, \ldots, 2n-1\}$ such that vertex $m+1$ does not receive color n. We now want to ensure that B_{m+1} is colored with $\{n+1,\ldots,2n-1\}$, i.e., does not use color n. This will involve uncoloring some vertices. Let

$$B' = \{v \mid v \text{ is colored } n \text{ and } v \text{ is adjacent to a non-colored vertex}\}.$$

We uncolor all the vertices in B'. Note that $B' \cap \{1,\ldots,m+1\} = \emptyset$, so (1) holds. Note that all colored neighbors of B' use colors $\{n+1,\ldots,2n-1\}$, hence all vertices in B_{m+1} use only these colors, so (2) holds. ▮

We will see later (Theorem 5.30) that the upper bound cannot be improved for general highly recursive graphs. However, if G is connected and $\chi(G) = 2$ then it is easy to see that $\chi^r(G) = 2$. This holds for both G recursive and G highly recursive.

5.5 Recursive Analogue is False for Highly Recursive Graphs

Theorem 5.21 gives an upper bound on the number of colors required to color an n-colorable highly recursive graph. The question arises 'Can we do better?' We cannot! That is, there exists a highly recursive graph G such that $\chi(G) = n$ and $\chi^r(G) = 2n-1$. The proof requires several definitions and lemmas. It was first proven in [145] but our exposition is based on a modification which was presented in [13].

Definition 5.22 Let $n \geq 3$. Let $G^n = (V,E)$ where

$$\begin{aligned} V = &\ \{(i,j) : 1 \leq i,j \leq n\} \\ E = &\ \{\{(i,j),(r,s)\} : i \neq r \text{ and } j \neq s\}. \end{aligned}$$

If $1 \leq i \leq n$, then the set of vertices $\{(i,j) : 1 \leq j \leq n\}$ is called the i^{th} *column of* G^n. The j^{th} *row of* G^n is defined similarly. The *basic row coloring* of G^n assigns color i to every vertex in the i^{th} row. The *basic column coloring* of G^n assigns color i to every vertex in the i^{th} column. Note that both are valid vertex colorings of G^n using only n colors.

Definition 5.23 If χ is a coloring of G^n, then χ *induces a colorful column (row)* if χ assigns to each vertex in a particular column (row) a different color. If the coloring being referred to is obvious, we may say 'G has a colorful column (row)' to mean that the coloring induces a colorful column (row).

Lemma 5.24 *Let χ be a coloring of G^n. Let $1 \leq i, j \leq n$ with $i \neq j$.*

1. *If color a appears more than once in row i (column i), then a can not appear in row j (column j).*
2. *If color a appears more than once in row i, and color b appears more than once in column j, then $a \neq b$.*

Proof:

1. Assume $\chi((x,i)) = \chi((y,i)) = a$. Every vertex (z,j) is connected to either (x,i), or (y,i), or both, hence $\chi((z,j)) \neq a$. Similar for the column version.
2. Assume $\chi((x,i)) = \chi((y,i)) = a$ and $\chi((j,w)) = \chi((j,z)) = b$. Since $x \neq y$, one of $\{x,y\}$ is not j; assume $x \neq j$. Since $w \neq z$, one of $\{w,z\}$ is not i. If $z \neq i$, then (x,i) and (j,z) are connected, so $a \neq b$. If $w \neq i$, then (x,i) and (j,w) are connected, so $a \neq b$.

▮

Lemma 5.25 *If χ is a $(2n-2)$-coloring of G^n, then χ either induces a colorful row or induces a colorful column, but not both.*

Proof: Assume, by way of contradiction, that χ is a $(2n-2)$-coloring of G^n that induces neither a colorful row nor column. For $1 \leq i \leq n$ let a_i (b_i) be a color that appears more than once in row i (column i).

By Lemma 5.24, (1) for all i, j with $i \neq j$, $a_i \neq a_j$ and $b_i \neq b_j$, and (2) for all i, j $a_i \neq b_j$. Hence the set $\{a_1, \ldots, a_n, b_1, \ldots, b_n\}$ has $2n$ colors, which contradicts χ being a $(2n-2)$-coloring. Hence χ induces a colorful row or column.

Assume, by way of contradiction, that χ induces a colorful row and a colorful column. Let row i and column j be colorful. The only way a vertex (k,i) in the ith row could have the same color as a vertex (j,m) in the jth column is if they are not connected, i.e., $k = j$ or $i = m$. If only one of these holds, then (k,i) and (j,m) are in the same row or column, and are colored differently; hence we must have $k = j$ and $i = m$. This means they are the same vertex. This happens exactly once, so the colorful row and column use a total of $2n-1$ colors. This contradicts χ being a $(2n-2)$-coloring.

▮

We now define a way to connect two graphs such that if in some coloring one of them has a colorful row (column) the other will have a colorful column (row).

Definition 5.26 Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs such that $G_1 \cong G_2 \cong G^n$. Assume the vertices of G_k are of the form (k, i, j) in such a way that (k, i, j) corresponds to (i, j). The following graph is the *2-element chain of* G_1 *and* G_2, denoted $CH(G_1, G_2)$.

$$\begin{aligned} V = &\ V_1 \cup V_2 \\ E = &\ E_1 \cup E_2 \cup E_{12} \\ E_{12} = &\ \{\{(1,i,j),(2,r,s)\} : i \neq s \text{ and } r \neq j\}. \end{aligned}$$

The edges in E_{12} are said to *link together* G_1 *and* G_2. Let $G_1, \ldots, G_s$ be graphs of type G^n. The *s-element chain of* $G_1, \ldots, G_s$, denoted by the expression $CH(G_1, \ldots, G_s)$, can be defined by linking G_1 to G_2, G_2 to G_3, ..., G_{s-1} to G_s.

In $CH(G_1, G_2)$ the r^{th} row of G_2 acts like the r^{th} column of G_1 in terms of which vertices of G_1 it is connected to. This intuition underlies the next lemma.

Lemma 5.27 *Let χ be a $2n-2$ partial coloring of $CH(G_1, G_2)$ that induces a colorful row (column) of the G_1 part. Any extension of χ to a $(2n-2)$-coloring of $CH(G_1, G_2)$ must induce a colorful column (row) in the G_2 part.*

Proof: Let χ and i be such that χ is a $2n-2$ partial coloring of $CH(G_1, G_2)$ that induces the i^{th} column of G_1 to be colorful. Assume, by way of contradiction, that there exists χ_0 such that χ_0 is a $(2n-2)$-coloring of $CH(G_1, G_2)$ which is an extension of χ but does not induce a colorful row of G_2. Every row of G_2 has some color repeated at least twice. For $1 \leq j \leq n$ let $c_j = \chi_0((1, i, j))$. For $1 \leq s \leq n$ let d_s be the color that, using χ, appears twice in the s^{th} row of G_2.

We show that $|\{c_1, \ldots, c_n, d_1, \ldots, d_n\}| > 2n - 2$. Since the i^{th} row was colorful, all the c_j's are distinct. By Lemma 5.24 all the d_s's are distinct. To show $|\{c_1, \ldots, c_n, d_1, \ldots, d_n\}| > 2n - 2$, we show that the only element that may be in $\{c_1, \ldots, c_n\} \cap \{d_1, \ldots, d_n\}$ is d_i.

Assume $c_j = d_s$. Let r_1 and r_2 be such that $\chi((2, r_1, s)) = \chi((2, r_2, s)) = d_s$. By definition $\chi((1, i, j)) = c_j$. Since $c_j = d_s$ there is no edge between $(2, r_1, s)$ and $(1, i, j)$; hence either $r_1 = j$ or $s = i$. Similarly, there is no edge between $(2, r_2, s)$ and $(1, i, j)$; hence either $r_2 = j$ or $s = i$. If $s \neq i$ then $r_1 = j$ and $r_2 = j$, so $r_1 = r_2$, which is false. Hence $s = i$. Therefore the only element of $\{c_1, \ldots, c_n\} \cap \{d_1, \ldots, d_n\}$ is d_i.
∎

Lemma 5.28 *Let χ be a $2n-2$ partial coloring of $CH(G_1, \ldots, G_s)$ that induces a colorful row (column) of the G_1 part. If s is even, then any extension χ_0 of χ to a $(2n-2)$-coloring of $CH(G_1, \ldots, G_s)$ must induce a colorful column (row) of the G_s part; if s is odd, then χ_0 must induce a colorful row (column).*

Proof: This follows from the previous lemma and induction. ∎

Lemma 5.29 *The graph $CH(G_1, \ldots, G_s)$ is n-colorable.*

Proof: For i even, color G_i by coloring every vertex in row j with color j. For i odd, color G_i by coloring every vertex in column j with color j. This is easily seen to be an n-coloring of $CH(G_1, \ldots, G_s)$. ∎

Theorem 5.30 *Let $n \geq 3$. There exists a highly recursive graph $\tilde{G}$ such that $\chi(\tilde{G}) = n$ and $\chi^r(\tilde{G}) = 2n-1$.*

Proof: We construct a highly recursive graph $\tilde{G}$ to satisfy the following requirements:

R_e : If $\{e\}$ is total, then $\{e\}$ is not a $(2n-2)$-coloring of $\tilde{G}$.

Recursively partition N into infinite sets $X_0, X_1, X_2, \ldots$. We satisfy R_e using vertices from X_e. Fix e. We show how to construct a highly recursive graph $G = G(e)$ such that (1) $\chi(G) = n$, and (2) $\{e\}$ is not a $(2n-2)$-coloring of G.

The graph $\tilde{G}$ is then simply $\bigcup_{e=0}^{\infty} G(e)$.

We construct $G = G(e)$ in stages. To avoid confusion we DO NOT use 'G_s', we merely speak of 'G at stage s.'

CONSTRUCTION of $G = G(e)$.

Stage 0: At this stage G consists of two graphs G_1 and G_2 such that $G_1 \cong G_2 \cong G^n$.

Stage s+1: (At the end of stage s, G consists of $CH(G_1, G_3, \ldots, G_{2s+1})$ and $CH(G_2, G_4, \ldots, G_{2s+2})$, where each G_i is isomorphic to G^n.) Run $\{e\}_s$ on all the vertices of G_1 and G_2. There are several cases.

Case 1: There exists a vertex in G_1 or G_2 where $\{e\}_s$ does not converge. Let G_{2s+3} and G_{2s+4} be graphs isomorphic to G^n that use the least numbers from X_e that are larger than s, but are not already in G, for vertices. Extend the $(s+1)$-chains to $(s+2)$-chains using G_{2s+3} for the odd chain, and G_{2s+4} for the even chain.

Case 2: $\{e\}_s$ converges on all the vertices of G_1 and G_2, and either uses more than $2n-2$ colors, or is not a coloring. Stop the construction of G since R_e is satisfied.

Case 3: $\{e\}_s$ converges on all the vertices in G_1 and G_2, uses at most $2n-2$ colors, is a coloring, and both G_1 and G_2 have colorful rows (columns). By Lemma 5.28 any extension of $\{e\}_s$ to a coloring of G will induce G_{2s+1} and G_{2s+2} to either both have a colorful column or both have a colorful row. We link G_{2s+1} to a (new) graph $H \cong G^n$, and then link G_{2s+2} to H (all new vertices are the least unused vertices of X_e). By Lemma 5.27, any extension of the coloring must induce both a colorful row and a colorful column in H. By Lemma 5.25, H cannot be $2n-2$ colored in this manner. Stop the construction, as R_e is satisfied.

Case 4: $\{e\}_s$ converges on all the vertices in G_1 and G_2, and is a $(2n-2)$-coloring of both G_1 and G_2; and this coloring induces G_1 to have a colorful row (column), and G_2 to have a colorful column (row). Link both G_{2s+1} and G_{2s+2} to a graph isomorphic to G^n. The coloring $\{e\}_s$ cannot be extended to a $(2n-2)$-coloring of G, since in such a coloring the new G^n graph would have to have both a colorful row and a colorful column.

END OF CONSTRUCTION

We show that the graph $G(e)$ is highly recursive. The vertex set is recursive since v is a vertex iff $v \in X_e$ and v was placed into the graph at some stage $s \leq v$. To determine the neighbors of a vertex v note that if v enters the graph at stage s then all the neighbors of v will enter by stage $s+1$.

Since $G(e)$ is highly recursive, it follows from Theorem 5.21 that $\chi^r(G) \leq$

$2n-1$ (this can also be proven directly). By the comments made before and during the construction it is easy to see that $\tilde{G}$ (the union of all the G's) is not recursively $(2n-2)$-colorable. Hence $\chi^r(\tilde{G}) = 2n-1$.

By Lemma 5.29 $\chi(\tilde{G}) = n$.

∎

5.6 Recursion-Theoretic Modification

By Theorems 5.15 and 5.30, n-colorable recursive and highly recursive graphs need not be recursively n-colorable. But such graphs do have n-colorings of low degree.

The following theorem is due to Bean [10].

Theorem 5.31 *If G is a recursive graph that is n-colorable, then there exists an n-coloring c that is of low degree.*

Proof: Let G be as in the hypothesis. Consider the following recursive n-ary tree T. The vertex $\sigma = (a_1, \ldots, a_m)$ (where for all i, $1 \leq a_i \leq n$) is on the tree T iff it represents a non-contradictory coloring (i.e., the partial coloring of G that colors vertex i with a_i does not label two adjacent vertices with the same color). We have (1) T is recursive, (2) T is recursively bounded by the function $f(m) = \langle n, \ldots, n\rangle$ (the n's appear m times), (3) any infinite branch of T is an n-coloring of G, and (4) every n-coloring of G is represented by some infinite branch of T (note that the set of infinite branches is nonempty since G is n-colorable). Since the infinite branches of T form a nonempty recursively bounded Π^0_1 class, by Theorem 3.12 there exists an infinite low branch. Since every infinite branch is an n-coloring, the theorem follows. ∎

The proof of the above theorem actually shows that the set of n-colorings of G is a recursively bounded Π^0_1 class. Remmel [139] has shown the converse: for any recursively bounded Π^0_1 class $\mathcal{C}$ and any constant $n \geq 3$, there exists a highly recursive graph G such that (up to a permutation of colors) there is an effective 1:1 degree-preserving correspondence between n-colorings of G and elements of $\mathcal{C}$.

5.7 Miscellaneous

We state several results about recursive-graph colorings without proof.

5.7.1 Bounding the Genus

If the genus of a (finite or infinite) graph G is bounded by g, then $\chi(G)$ is bounded by a function of g which we denote $c(g)$. In 1890 Heawood[79] showed that, for $g \geq 1$, $c(g) \leq \lfloor \frac{7+\sqrt{48g+1}}{2} \rfloor$. In 1967 Ringel and Young [140] proved that this bound is tight. (See Chapter 5 of [69] for proofs of both the upper and lower bound). Appel and Haken [4, 5] showed that $c(0) = 4$ (e.g., planar graphs are 4-colorable) using very different techniques which involved a rather long computer search.

We wonder if an analogue of $c(g)$ exists for recursive or highly recursive graphs (i.e., perhaps every recursive graph of genus g is $c'(g)$-colorable for some c'). Since the graph G constructed in Corollary 5.16 is planar (i.e., its genus is 0) and $\chi^r(G) = \infty$, no analogue of $c(g)$ exists for recursive colorings of recursive graphs. However, for highly recursive graphs G, Bean[10] showed that $\chi^r(G) \leq 2(c(g) - 1)$. Using the 4-color Theorem this yields that if G is planar then $\chi^r(G) \leq 6$ (Bean[10] obtained this result without using the 4-color Theorem). Using the 4-color Theorem Carstens [26, 28] claims to have shown[1] that if G is a highly recursive planar graph then $\chi^r(G) \leq 5$. It is an open problem to obtain $\chi^r(G) \leq 4$. More generally, it is an open problem to improve Bean's bound for general genus g, or show it cannot be improved. It may be of interest to impose additional recursion-theoretic constraints such as having a recursive embedding on a surface of genus g.

5.7.2 Bounding the degree

The *degree* of a graph is the maximal degree of a vertex. A graph G *satisfies* Δ_d if it has degree d and does not have a subgraph isomorphic to K_{d+1} (the complete graph on $d + 1$ vertices).

Brooks[21, 22] showed that if a (finite or infinite) graph G satisfies Δ_d then $\chi(G) \leq d$. By a variation of Theorem 5.15, for every d there exists a recursive graph G that satisfies Δ_d but $\chi^r(G) = d + 1$; hence the recursive analogue of Brooks's theorem fails for recursive graphs. Schmerl[146] showed that the recursive analogue does hold for highly recursive graphs. Carstens and Pappinghaus[30] discovered the result independently, and Tverberg[164]

[1] The paper sketches the proof and promises further work with details that, to our knowledge, has not appeared.

has given a simpler proof. It is an open question as to just how wide the gap between $\chi(G)$ and $\chi^r(G)$ may be for recursive graphs with property Δ_d.

5.7.3 Regular Graphs

A graph is d-regular if every vertex has degree d. Note that a recursive d-regular graph is highly recursive. Schmerl[145] posed the following question: if $2 \leq n \leq m \leq 2n-2$, what is the least d for which there is a recursive d-regular graph which is not recursively m-colorable? We denote this quantity by $d(n,m)$.

Schmerl [145] notes that if there exists a highly recursive G with degree bound d then there exists a highly recursive d-regular G' such that $\chi(G) = \chi(G')$ and $\chi^r(G) = \chi^r(G')$. Bean constructed, for every $k \geq 2$, a highly recursive G with $\chi(G) = k$, $\chi^r(G) = k+1$, and degree bound $2k-2$. Hence $d(n,n+1) \leq 2n-2$ (Manaster and Rosenstein [119] obtained the same result with different methods). Schmerl showed how to modify Bean's construction to obtain $d(n,n+1) \leq \left\lfloor \frac{3n-1}{2} \right\rfloor$. Since the degree bound of the graph constructed in Theorem 5.30 (of this survey) is $3(n-1)^2$, $d(n,2n-2) \leq 3(n-1)^2$.

5.7.4 Perfect Graphs

It is of interest to impose graph-theoretic conditions on a highly recursive graph G such that if G satisfies the condition then $\chi^r(G)$ is not too far from $\chi(G)$. A graph is *perfect* if for every induced subgraph H, $\omega(H) = \chi(H)$ (where $\omega(H)$ is the size of the largest clique in H). Kierstead[95] proved that if G is a highly recursive perfect graph then $\chi^r(G) \leq \chi(G)+1$. It is a (vague) open question to find other graph-theoretic conditions that narrow the gap between $\chi(G)$ and $\chi^r(G)$.

5.7.5 On-line colorings

Informally, an on-line algorithm to color an (infinite) graph is an algorithm that is given the graph a vertex at a time, and has to color a vertex as soon as it sees it. For general graphs it is hopeless to try to bound the number of colors such an algorithm will need to use, but we can bound the number of colors it uses on the first n vertices. It is trivial to color the first

n vertices with n colors. Lovász, Saks, and Trotter [117] have improved this by showing (1) if $\chi(G) \leq 2$ then G can be colored on-line via an algorithm that uses $2 \log n$ colors on the first n vertices; (2) if $\chi(G) \leq k$ then G can be colored on-line via an algorithm that uses $O(n \frac{\log^{(2k-3)} n}{\log^{(2k-4)} n})$ on the first n vertices ($\log^{(m)} n$ is iterated log). There are limits on the extent to which these bounds can be improved: Vishwanathan [166] showed that for every on-line algorithm, and every k and n, there exists a graph G on n vertices (and an order to present it) so that $\chi(G) \leq k$ and the algorithm must use at least $(\frac{\log n}{4k})^{k-1}$ colors. Irani [86] has shown that certain classes of graphs (which include planar graphs) have a presentation with which they are on-line colorable with $O(\log n)$ colors. For a survey of this area see [98, 103].

There are connections between on-line coloring algorithms and combinatorial analogues of Dilworth's Theorem. See Section 7.4.2 for an overview.

5.7.6 Coloring Directed Graphs

More complex conditions can be imposed on directed graphs than on undirected graphs. Kierstead [97] has found one such condition that affects the recursive chromatic number. He has shown that if G is a recursive directed graph that does not have an induced subgraph of the form (1) directed 3 cycle, or (2) $\circ \longrightarrow \circ \longrightarrow \circ \longleftarrow \circ$, or (3) $\circ \longleftarrow \circ \longrightarrow \circ \longrightarrow \circ$, then $\chi^r(G) \leq 2^{\omega(G)}$ where $\omega(G)$ is the size of the largest clique.

5.7.7 Coloring Interval Graphs

An *interval graph* is the comparison graph for an interval order (see Section 7.7.1 for the definition of an interval order). Kierstead and Trotter [102] have shown that recursive interval graphs are $(3\omega(G) - 2)$-colorable, where $\omega(G)$ is the size of the largest clique in G.

5.7.8 Decidable Graphs

Bean[10] considered imposing stronger recursive conditions on a graph than highly recursive

Definition 5.32 A graph G is *decidable* if there is a decision procedure to determine if a given first order sentence about it is true. The language in which the sentences are expressed has (1) the usual logic symbols including quantifiers that range over vertices, (2) the symbol $E(x, y)$ (tests if x and y are connected by an edge), and the symbol '=' for equality.

Bean showed that all negative results that he obtained for highly recursive graphs also hold for decidable graphs. Combining his technique with Theorem 5.30 yields that for every $k \geq 2$ there exists a decidable graph G such that $\chi(G) = k$ and $\chi^r(G) = 2k - 1$. That construction can be combined with the recursion-theoretic techniques of Theorem 5.20 to obtain that the set of indices of decidable graphs G such that $\chi(G) = k$ and $\chi^r(G) = 2k - 1$ is Σ_3-complete.

It is an open question to find a reasonable recursive condition for graphs G that implies $\chi(G) = \chi^r(G)$. Expanding the language in which the sentences are expressed may help. A comprehensive study of types of decidable graphs has not been undertaken.

Dekker[46] examined graphs where one can decide whether 2 vertices are connected by a path, but he did not examine coloring.

5.7.9 A-recursive Graphs

Gasarch and Lee [65] considered graphs that were intermediary between recursive and highly recursive. Let nbd_G be the function that, on input x (a vertex of G) outputs all the neighbors of G. Note that if G is recursive then $nbd_G \leq_T K$ and if G is highly recursive then $nbd_G \leq_T A$.

Definition 5.33 Let A be any set. A graph $G = (V, E)$ is *A-recursive* if G is recursive and $nbd(G) \leq_T A$.

A natural question is to see, for various sets A with G being A-recursive implies any finite bound on $\chi^r(G)$. The answer is no:

Theorem 5.34 *Let A be a non-recursive r.e. set. There exists an A-recursive graph G such that G is 2-colorable but not recursively k-colorable for any natural number k.*

The proof is a variant of Bean's original construction with a permitting argument. The former enables us to show that the graph is 2-colorable but not recursively k-colorable for any k. The latter allows us to show that the neighbor function is recursive in the r.e. set A. For a discussion of the permitting method, see [159]. It is an open question to extend the theorem to all A such that $\emptyset <_{\mathrm{T}} A <_{\mathrm{T}} K$. Even the case where A is 2-r.e. is open.

The proof technique can be extended to show the following generalization of Theorem 5.15.

Theorem 5.35 *Let A be a non-recursive r.e. set. Let a, b be such that $2 \le a < b \le \infty$. Let X be an infinite recursive set. There exists an A-recursive graph $G = (V, E)$ such that $\chi(G) = a$, $\chi^r(G) = b$, and $V \subseteq X$. If $a \le 4$ then G can be taken to be planar.*

5.7.10 Complexity of Finding $\chi(G)$ and $\chi^r(G)$

Theorem 5.20 says that determining $\chi^r(G)$ will *require* a $\emptyset'''$ oracle. A comprehensive study of *how many queries* are required to determine $\chi(G)$ and $\chi^r(G)$ was undertaken by Beigel and Gasarch [13, 14]. In those papers 64 questions were raised (six 2-valued parameters were varied), of which 58 were solved exactly. We present two theorems that encompass four of these questions.

Theorem 5.36 *Let $c \ge 2$. Let D_c (D_c^r) be the set of indices of recursive graphs G such that $\chi(G) \le c$ ($\chi^r(G) \le c$). Let χ_c and χ_c^r be the partial functions*

$$\chi_c(e) = \begin{cases} \chi(G_e^r) & \text{if } e \in D_c; \\ \uparrow & \text{if } e \notin D_c. \end{cases}$$

$$\chi_c^r(e) = \begin{cases} \chi^r(G_e^r) & \text{if } e \in D_c^r; \\ \uparrow & \text{if } e \notin D_c^r. \end{cases}$$

There is a solution to the promise problem (D_c, χ_c) that can be computed with $\lceil \log(c+1) \rceil$ queries to K. For every set X, no solution to (D_c, χ_c) can be computed with $\lceil \log(c+1) \rceil - 1$ queries to X. If X is any set such that $K \not\le_T X$ then (D_c, χ_c) cannot be computed with X. Similar theorems hold for computing (D_c^r, χ_c^r) with oracle $\emptyset'''$. Similar theorems hold for highly recursive graphs.

Theorem 5.37 *Let f and g be recursive functions such that (1) $\sum_{i=0}^{\infty} 2^{-f(i)} \le 1$ and is an effectively computable real r (i.e., there exists a recursive function $h : Q \to Q$ such that $|h(\epsilon) - r| < \epsilon$), and (2) $\sum_{i=0}^{\infty} 2^{-g(i)} > 1$. Let D*

be the set of valid indices for recursive graphs. There is a solution for the promise problem (D, χ) that, on input e, takes $f(\chi(G_e))$ queries to K. For every set X, no solution to (D, χ) can be computed with $g(\chi(G_e))$ queries to X. Similar results hold for χ^r with a $\emptyset'''$ oracle. Similar results hold for highly recursive graphs.

5.7.11 Actually finding a coloring

None of the results looked at so far involve actually coloring the graph. Beigel and Gasarch[16] examined this issue in terms of the number of times a recursive procedure will have to change its mind while coloring a graph. They constructed graphs where a recursive mapmaker has to recolor the map many times.

Definition 5.38 Let $G = (V, E)$ be a k-colorable recursive graph. A *local k-coloring of G* is a function that takes a finite set $H \subseteq V$ and outputs a k-coloring of H that is extendible to a k-coloring of all of G.

We examine the complexity of local k-colorings. Our measure of complexity is 'mind-changes.' In particular we study algorithms for local k-colorings that are allowed to change their mind $g(n)$ times on inputs consisting of n vertices. The function g is the complexity of the algorithm. There are recursive graphs for which every local coloring changes its mind many times.

In what follows we will interpret the input to a Turing machine as an ordered pair (H, s) where H is a finite set of vertices and s is a parameter; and the output as a coloring of those vertices.

Definition 5.39 Let f be a function from $[\mathsf{N}]^{<\omega}$ to N, and let g be a function from N to N. The function f is *computable by a g-mind-change algorithm* if there exists a total Turing machine M such that, for every $H \in [\mathsf{N}]^n$ (1) $\lim_{s\to\infty} M(H, s) = f(H)$ (i.e., $(\exists s_0)(\forall s \geq s_0) M(H, s) = f(H)$), and (2) $|\{s : M(H, s) \neq M(H, s+1)\}| \leq g(n)$.

Carstens and Pappinghaus[30] showed that one can color any recursive graph with a mind-change algorithm that changes its mind an exponential number of times. We sharpen their result and put it in our terminology. Let $NI(n, k)$ be the number of nonisomorphic colorings of n vertices with k colors. It can be shown that $NI(n, k) = \sum_{t=0}^{k} \frac{t^n}{t!} \sum_{r=0}^{k-t} \frac{(-1)^r}{r!}$. For large n and fixed k this is approximately $k^n/k!$.

Theorem 5.40 *Let $k \geq 3$. Let $G = (V, E)$ be a k-colorable recursive graph. There exists a local k-coloring of G that is computable by a g-mind-change algorithm where $g(n) = NI(n, k) - 1$. There exists a k-colorable recursive graph G such that every mind-change algorithm that computes a local k-coloring of G requires $NI(n, k) - 1$ mind-changes on an infinite number of inputs H of arbitrarily large cardinality.*

5.7.12 Polynomial Graphs

Cenzer and Remmel [36] have considered graphs with labels in $\{0, 1\}^*$ such that testing for an edge can be done in polynomial time. They have shown the following.

Theorem 5.41

1. *If G is a recursive graph and $k \in \mathsf{N}$ then there exists a poly graph G' such that there is an effective degree-preserving map from the k-colorings of G to the k-colorings of G. Hence, using Theorem 5.15, there exists a poly graph that is 3-colorable but not recursively k-colorable for any k.*

2. *There exists a poly graph G that is 2-colorable, connected, but not primitive-recursively 2-colorable. This is of interest since it shows that the natural analog of Note 5.17.2 is false.*

6 Hall's Theorem on Bipartite Graphs

We consider the infinite version of Hall's Theorem on solutions to bipartite graphs. We (1) present the finite and infinite versions (due to Phillip Hall [75, 76] and Marshall Hall [74] respectively), (2) show that a recursive analogue of Hall's Theorem is false, (3) show that a recursion-theoretic modification is true, (4) show that there is a modification that is both recursion-theoretic and combinatorial which is true, and, (5) state some miscellaneous results.

Hall's theorem for finite graphs also yields an algorithm for testing if a bipartite graph has a solution, and, if so, finding it. These algorithms are not efficient. See [131] or [58] for efficient algorithms for these problems.

6.1 Definitions and Classical Version

Definition 6.1 A *bipartite graph* G is a 3-tuple (A, B, E) where A and B are disjoint sets of vertices and $E \subseteq [A \cup B]^2 - ([A]^2 \cup [B]^2)$ (i.e., E consists of unordered pairs of vertices, one from A and one from B). If $\forall x \in A \cup B$, $degree(x) < \infty$, then we say G has *finite degree*. Henceforth G has finite degree. The neighbors of a finite set of vertices $X \subseteq A$ are denoted $nb_G(X)$. Formally we define $nb_G : P^{<\omega}(A) \to P^{<\omega}(B)$ as follows: for each finite $X \subseteq A$, $nb_G(X) = \{b \in B \mid (\exists a \in X)\{a, b\} \in E\}$. Note that from the function nb_G one can obtain all the edges of G. When G is clear from context we abbreviate nb_G by nb.

Definition 6.2 Let $G = (A, B, E)$ be a bipartite graph. A function $f : A \to B$ is a *solution* for G if f is one to one and $\forall a \in A$ $\{a, f(a)\} \in E$. Given $X \subseteq A$ and $Y \subseteq B$, we will sometimes call $f : X \to Y$ a *solution from* X *to* Y. If f is onto then the solution is *symmetric*.

We will be considering the infinite version of Hall's Theorem. We present the finite and infinite versions. The proof we give for the finite case does not lead to a computationally efficient algorithm to find a solution. The most efficient algorithm known for this problem runs in time $O(|V|^{\frac{1}{2}}|E|^2)$ (see [131, p. 226]).

Definition 6.3 Let $G = (A, B, E)$ be a (finite or infinite) bipartite graph. G satisfies *Hall's condition* if for all finite $X \subseteq A$, $|nb_G(X)| \geq |X|$.

Theorem 6.4 (Finite Hall's Theorem) *Suppose $G = (A, B, E)$ is a finite bipartite graph. Then G has a solution iff G satisfies Hall's condition.*

Proof: If G does not satisfy Hall's condition, then there is an $X \subseteq A$ such that $|nb_G(X)| < |X|$. Obviously, there is no solution for X, so there can be no solution for G.

Now suppose G satisfies $\forall X \subseteq A$, $|X| \leq |nb_G(X)|$. Let $n = |A|$. We will prove by induction on n that there is a solution for G.

If $n = 1$, let $A = \{a\}$. Since $|nb_G(\{a\})| \geq 1$, there is $b \in B$ such that $\{a, b\} \in E$. Then $M = \{(a, b)\}$ is a solution for G.

If $n > 1$, assume the theorem holds for bipartite graphs (A, B, E) with $|A| < n$. We will consider two cases:

Case 1: Suppose for all k with $1 \leq k < n$ for all $X \subseteq A$ with $|X| = k$, $|nb_G(X)| \geq k + 1$. Then choose any $\{a, b\} \in E$ with $a \in A$. Let G' be $\langle A - \{a\}, B - \{b\}, E' \rangle$ where E' is E restricted to edges that do not involve a or b. Note that for all finite subsets $X \subseteq A - \{a\}$ with $|X| = k$ we have $|nb_{G'}(X)| \geq k$. By our induction hypothesis G' has a solution M. Then $M \cup (a, b)$ is a solution for G.

Case 2: Suppose there is an $X \subset A$ and $k < n$ such that $|X| = |nb_G(X)| = k$. Let $G' = \langle X, B, E' \rangle$ where E' is the set of edges between elements of X and elements of B. By our induction hypothesis, G' has a solution M. Since $|X| = |nb_G(X)|$ and solutions are one to one, $image(M) = nb_G(X)$.

Now we need to show there is a solution from $A - X$ to $B - nb_G(X)$. Let $G'' = \langle A - X, B - nb_G(X), E'' \rangle$ where E'' is the subset of E consisting of pairs of elements $\{x, y\}$ such that $x \in A - X$, $y \in B - nb_G(X)$. Assume, by way of contradiction, that there exists $C \subseteq A - X$ such that $|nb_{G''}(C)| < |C|$. Then

$$\begin{aligned} nb_G(C) = & \ (nb_G(C) \cap nb_G(X)) \cup (nb_G(C) \cap \overline{nb_G(X)}) \\ = & \ (nb_G(C) \cap nb_G(X)) \cup nb_{G''}(C). \end{aligned}$$

Hence

$$nb_G(C \cup X) = nb_G(C) \cup nb_G(X) = nb_{G''}(C) \cup nb_G(X).$$

We can now show that $|nb_G(C \cup X)| < |C \cup X|$, which contradicts that G satisfies Hall's condition.

$$|nb_G(C \cup X)| = |nb_{G''}(C)| + |nb_G(X)| = |nb_{G''}(C)| + |X| < |C| + |X| = |C \cup X|,$$

which contradicts our hypothesis.

Then by our induction hypothesis, there is a solution M' from $A - X$ to $B - nb_G(X)$. Then $M \cup M'$ is a solution for G. ∎

We now prove the infinite Hall's Theorem. We give a direct proof; it can also be proven by König's Lemma (Theorem 3.3).

Theorem 6.5 *(Infinite Hall's Theorem) Suppose $G = (A, B, E)$ is a countable bipartite graph with finite degree. Then G has a solution iff G satisfies Hall's condition.*

Proof: If G does not satisfy Hall's condition then there is some finite $X \subseteq A$ such that $|nb_G(X)| < |X|$. Obviously, there is no solution for X, so there can be no solution for G.

Now suppose G satisfies Hall's condition. Since A is countable, let $A = \{a_1 < a_2 < \cdots\}$. Given $n \in \mathsf{N}$, let $A^n = \{a_0, \ldots, a_n\}$, $B^n = nb_G(A^n)$, $E^n = E \cap \{\{a,b\} \mid a \in A^n, b \in B^n\}$, and $G^n = (A^n, B^n, E^n)$. For $n \in \mathsf{N}$, G^n satisfies Hall's condition, so by the finite Hall's Theorem, there is a solution M^n for G^n. We will build a solution for G from the M^n. Let

$M(a_1) = \mu x[(\exists^\infty s) M^s(a_1) = x]$

$M(a_{n+1}) = \mu x[(\exists^\infty s)(\bigwedge_{j=1}^{n}(M^s(a_j) = M(a_j)) \wedge M^s(a_{n+1}) = x)]$

It is easy to see that M is a solution. ▮

The proof of Theorem 6.5 given above is noneffective. To see if the proof could have been made effective we will look at a potential analogue. In order to state this analogue we need some definitions.

Definition 6.6 Let $G = (A, B, E)$ be a bipartite graph with $A, B \subseteq \mathsf{N}$. G is a *recursive bipartite graph* if A, B and E are recursive, and G has finite degree. Note that a recursive bipartite graph is different from a bipartite recursive graph. G is a *highly recursive bipartite graph* if G is recursive and the function nb_G is recursive. Note that a highly recursive bipartite graph is different from a bipartite highly recursive graph.

We will use the recursive tripling function to represent recursive and highly recursive bipartite graphs.

Definition 6.7 A number $e = \langle e_1, e_2, e_3 \rangle$ *determines a recursive bipartite graph* if $e_1, e_2 \in TOT01$, $e_3 \in TOT$, and the sets $A = \{a \mid \{e_1\}(a) = 1\}$ and $B = \{b \mid \{e_2\}(b) = 1\}$ are disjoint. The *recursive bipartite graph determined by* e is (A, B, E) where $E = \{\{a,b\} \mid a \in A, b \in B, \{e_3\}(a,b) = \{e_3\}(b,a) = 1\}$.

Definition 6.8 A number $e = \langle e_1, e_2, e_3 \rangle$ *determines a highly recursive bipartite graph* if $e_1, e_2 \in TOT01$ $e_3 \in TOT$, and the sets $A = \{a \mid \{e_1\}(a) = 1\}$ and $B = \{b \mid \{e_2\}(b) = 1\}$ are disjoint. The *recursive bipartite graph determined by* e is (A, B, E) where E is determined by the fact that $\{e_3\}$ computes nb_G. (Here, $\{e_3\}$ is interpreted as a function from N to finite subsets of N.)

Definition 6.9 Let $G = (A, B, E)$ be a bipartite graph such that $A, B \subseteq \mathsf{N}$ (in practice G will be a recursive or highly recursive bipartite graph). A function $f : \mathsf{N} \to B$ is a *recursive solution* for G if f is total recursive and f, when restricted to A, is a solution for G.

Potential Analogue 6.10 There is a recursive algorithm $\mathcal{A}$ that performs the following. Given an index e for a highly recursive bipartite graph that satisfies Hall's condition, $\mathcal{A}$ outputs an index for a recursive solution. A consequence is that if a highly recursive bipartite graph G has a solution, then G has a recursive solution.

We will soon see (Theorem 6.13) that this potential analogue is false. We will then have a recursion-theoretic modification which is true. No combinatorial analogue appears to be true; however, we will then impose combinatorial *and* recursion-theoretic conditions that will yield a true analogue.

6.2 Recursive Analogue is False

The following theorem is due to Manaster and Rosenstein [118].

Definition 6.11 If $v_1, v_2, \ldots, v_n \in \mathsf{N}$, and all the v_i are distinct, then *the line graph of* $(v_1, \ldots, v_n)$ is the graph $G = (V, E)$ where $V = \{v_1, \ldots, v_n\}$ and $E = \{\{v_i, v_{i+1}\} : 1 \le i \le n-1\}$. The vertices $\{v_1, \ldots, v_i\}$ are *strictly to the left of* v_{i+1}. The vertex v_1 is the *left endpoint of* G. Terms using 'right' instead of 'left' can be defined similarly.

We would like to interpret line graphs as bipartite graphs. To do this we need to specify one vertex as being in A (or B), which will determine the status of the other vertices.

The following lemma is easy to prove, hence we leave it to the reader.

Lemma 6.12 *Let $G[i, j]$ be the line graph on*

$$(v_i, v_{i-1}, \ldots, v_1, x, y, z, w_1, w_2, \ldots, w_j).$$

Interpret $G[i, j]$ as being bipartite by assuming $y \in A$.

1. *If i is odd then (y, x) cannot be in any solution of A to B in G.*

2. *If j is odd then (y, z) cannot be in any solution of A to B in G.*

Proof:

We prove a; the proof for b is similar. We use induction on i.

Let $i = 1$. Note that $v_1 \in A$. If M is any solution of $G[1, j]$ then M must use (v_1, x), else v_1 (which is in A) will be unmatched. Hence M cannot use (y, x).

We assume the statement true for odd i and we prove it for $i + 2$. Note that $v_{i+2} \in A$. Let M be a solution of $G[i + 2, j]$. M must use (v_{i+2}, v_{i+1}), else v_{i+2} will be unmatched. Hence M cannot use (v_i, v_{i+1}). Therefore $M - \{(v_{i+2}, v_{i+1})\}$ is a solution of $G[i, j]$. By the induction hypothesis, M does not contain (y, x). ∎

Theorem 6.13 (Manaster and Rosenstein [118]) *There exists a highly recursive bipartite graph $G = (A, B, E)$ that satisfies Hall's condition but has no recursive solution.*

Proof:

We construct a highly recursive bipartite graph $G = (A, B, E)$ that has a solution (hence satisfies Hall's condition) and satisfies the following requirements.

R_e : $\{e\}$ total $\Rightarrow$ $\{e\}$ is not a solution of G.

Recursively partition N into infinite recursive sets $X_0, X_1, \ldots$. We construct a highly recursive bipartite graph $G(e) = (A(e), B(e), E(e))$ such that the following hold.

1. $A(e) \cup B(e) = X_e$.

2. $G(e)$ has a solution.

3. $\{e\}$ is not a solution of $G(e)$.

The union $G = \bigcup_{e \geq 0} G(e)$ is the desired graph.

In our description of $G(e)$, whenever we need a vertex we take the least unused vertex of X_e. We denote the bipartite graph constructed by the end of stage e by $G(e, s) = (A(e, s), B(e, s), E(e, s))$.

CONSTRUCTION

Stage 0: Let $G(e,0)$ be the line graph on (a,b,c) (three new vertices—the least three elements of X_e), interpreted as a bipartite graph by specifying $b \in A(e,0)$.

Stage $s+1$: (Assume inductively that $G(e,s)$ is a line graph.) Run $\{e\}_s(b)$. There are 4 cases.

1. If $\{e\}_s(b)\uparrow$ or $\{e\}_s(b)\downarrow\notin \{a,c\}$, then form $G(e,s+1)$ by adding one vertex to each end of $G(e,s)$.

2. If $\{e\}_{s-1}(b)\uparrow$ and $\{e\}_s(b)\downarrow = a$, then perform whichever of the following two cases applies. In all future stages t never place a vertex on the left end of $G(e,t)$ again.

    a) if there is an even number of vertices strictly to the left of a in $G(e,s)$, then $G(e,s+1)$ is formed by placing one vertex on each end of $G(e,s)$;

    b) if there is an odd number of vertices strictly to the left of a in $G(e,s)$, then $G(e,s+1)$ is formed by placing one vertex on the right end of $G(e,s)$.

3. If $\{e\}_{s-1}(b)\uparrow$ and $\{e\}_s(b)\downarrow = c$, then this is similar to case 2 except that we are concerned with the right side of $G(e,s)$ and $G(e,t)$.

4. If $\{e\}_{s-1}(b)\downarrow\in \{a,c\}$, then at a previous stage case 2 or 3 must have taken place. $G(e,s+1)$ is formed by adding a vertex to whichever end of $G(e,s)$ is permitted.

END OF CONSTRUCTION

$A(e)$ and $B(e)$ are both recursive: to determine if $p \in A(e)$ or $p \in B(e)$ either (1) $p \notin X_e$, so $p \notin A(e)$ and $p \notin B(e)$, or (2) $p \in X_e$, in which case run the construction until p appears in the graph, and note whether p enters in the A or B side.

$G(e)$ is highly recursive: if $p \in A(e) \cup B(e)$, then all the neighbors of p appear the stage after p itself appears.

Since $G(e)$ is just the 2-way or 1-way infinite line graph, it obviously has a solution.

We show that $\{e\}$ is not a solution of $G(e)$. If $\{e\}(b)\uparrow$ or $\{e\}(b)\downarrow\notin\{a,c\}$, then $\{e\}$ is clearly not a solution. If $\{e\}(b)\downarrow$, then case 2 or 3 will occur, at which point $\{e\}$ will be forced not to be a solution of $G(e)$, by Lemma 6.12. ∎

For recursive bipartite graphs the situation is even worse. Manaster and Rosenstein [118] have shown that there exist recursive bipartite graphs that satisfy Hall's condition but do not have any solution recursive in K. If we allow our bipartite graphs to have infinite degree then the situations is far worse. Misercque [126] has shown that for every recursive tree T there exists a recursive bipartite graph G such that there is a degree preserving bijection between the infinite branches of T and the solutions of G. Since there exists recursive trees T where every infinite branch is not hyperarithmetic [142, Page 419, Corollary XLI(b)], there is a recursive bipartite graph where every solution is not hyperarithmetic.

6.3 How Hard is it to Determine if there is a Recursive Solution?

By Theorem 6.13 there are highly recursive bipartite graphs that satisfy Hall's condition, but no recursive solution. We investigate how hard it is to determine if a particular highly recursive bipartite graph is of that type. By contrast the following promise problem is Π_1-complete: (D,A), where $D=\{e \mid e \text{ is the index of a highly recursive bipartite graph}\}$ and $A=\{e\in D \mid \text{ the graph represented by } e \text{ has a solution}\}$.

Notation 6.14 Let HRB be the set of valid indices of highly recursive bipartite graphs.

Theorem 6.15 *The set*

$$RECSOL=\{e : e\in HRB \wedge G_e \textit{ has a recursive solution}\}$$

is Σ_3-complete.

Proof:

For this proof, if $e = \langle e_1, e_2, e_3 \rangle$ is an index that determines a highly recursive bipartite graph, then we denote the graph that it determines by $G_e = (A_e, B_e, E_e)$. We abbreviate $(\exists t)\{e_1\}_t(x) = 1$ by $x \in A_e$, and adopt similar conventions for B_e, E_e.

$RECSOL$ is the set of all triples $e = \langle e_1, e_2, e_3 \rangle$ of numbers in TOT such that there exists an i such that

1. $i \in TOT$, and
2. $(\forall x, y)[(x \in A_e \wedge \{i\}(x) = y) \Rightarrow \{x, y\} \in E_e]$, and
3. $(\forall x, z)[(x, z \in A_e \wedge x \neq z) \Rightarrow \{i\}(x) \neq \{i\}(z)]$

Clearly $RECSOL$ is Σ_3. To show $RECSOL$ is Σ_3-hard we show that $COF \leq_m RECSOL$. Given x, we 'try' to construct a highly recursive bipartite graph G that satisfies Hall's condition but does not have a recursive solution. We will always succeed in making G satisfy Hall's condition. If $x \in COF$ then our attempt will fail in that G will have a recursive solution. If $x \notin COF$ then our attempt will succeed in that G will not have a recursive solution.

We try to construct a highly recursive bipartite graph $G = (A, B, E)$ that satisfies the following requirements.

$R_e : \{e\}$ total $\Rightarrow \{e\}$ is not a solution of G.

Recursively partition N into infinite recursive sets $X_0, X_1, \ldots$. We construct a highly recursive bipartite graph $G(e) = (A(e), B(e), E(e))$ such that the following hold.

1. $A(e) \cup B(e) = X_e$.
2. $G(e)$ has a solution.
3. If $\overline{W}_x \cap \{e, e+1, \ldots\} \neq \emptyset$, then $\{e\}$ is not a solution of $G(e)$.
4. If $\overline{W}_x \cap \{e, e+1, \ldots\} = \emptyset$, then $G(e)$ has a recursive solution.

The union $G = \bigcup_{e \geq 0} G(e)$ is the desired graph. If there is an e_0 such that $\overline{W}_x \cap \{e_0, e_0+1, \ldots\} = \emptyset$ then, for all $e \geq e_0$, $\overline{W}_x \cap \{e, e+1, \ldots\} = \emptyset$, hence

$G(e)$ will have a recursive solution. From this we will be able to deduce that G has a recursive solution.

In our description of $G(e)$, whenever we need a vertex we take the least unused number of X_e. We denote the bipartite graph constructed by the end of stage e by $G(e,s) = (A(e,s), B(e,s), E(e,s))$. $G(e)$ is itself a union of disjoint line graphs. During each stage of the construction we are adding vertices to one of those line graphs, which we refer to as 'the current component'.

CONSTRUCTION

Stage 0: Let $G(e,0)$ be the line graph on (a,b,c) (the least three elements of X_e), interpreted as a bipartite graph by specifying $b \in A(e,0)$. This will be the current component until it is explicitly changed. Set $\Gamma_{e,0} = e$.

Stage s+1: If $\Gamma_{e,s} \notin W_{x,s}$, then set $\Gamma_{e,s+1} = \Gamma_{e,s}$ and proceed to work on R_e using the current component and current values of a, b, and c, as in the construction in Theorem 6.13. If $\Gamma_{e,s} \in W_{x,s}$, then

1. Set $\Gamma_{e,s+1} = \mu z[z > \Gamma_{e,s} \wedge z \notin W_{x,s}]$.

2. If the number of vertices in the current component is odd then add a vertex to it (respecting whatever restraints there may be on which side you can add to).

3. Let a, b, c be three new vertices (the least unused numbers in X_e) and start a new current component with the line graph on (a,b,c), interpreted as a bipartite graph by taking $b \in A(e,s+1)$.

END OF CONSTRUCTION

Let $G = \bigcup_{e \geq 0} G(e)$. For each e the graph $G(e)$ is highly recursive by reasoning similar to that used in the proof of Theorem 6.13. It has a solution since it consists of some number (possibly infinite) of finite graphs with an even number of vertices, and at most one infinite graph which is either the infinite two-way line graph or the infinite one-way line graph. Since all the algorithms for all the $G(e)$ are uniform, G is a highly recursive graph with a solution.

If $x \notin COF$ then, for all e, $\overline{W}_x \cap \{e, e+1, \ldots\} \neq \emptyset$. Let y_e be the least element of $\overline{W}_x \cap \{e, e+1, \ldots\}$. Then $\lim_{s \to \infty} \Gamma_{e,s} = y_e$, and in particular it exists. Eventually the attempt to make sure $\{e\}$ is not a solution of $G(e)$

will be acting on one component. In this case such efforts will succeed, as in the proof of Theorem 6.13. Now consider G. This graph has no recursive solution since, for all e, $\{e\}$ is not a solution of $G(e)$.

If $x \in COF$, then we show that G has a recursive solution. For almost all e, $\overline{W}_x \cap \{e, e+1, \ldots\} = \emptyset$. Hence, for almost all e, $\lim_{s\to\infty} \Gamma_{e,s} = \infty$. Hence, for almost all e, all the components of $G(e)$ are finite and have an even number of vertices. Let S be the finite set of e such that $G(e)$ has an infinite component. For every $e \in S$ $G(e)$ has a finite number of finite components which have an even number of vertices, and one component that is either the two-way or one-way infinite line graph. Hence $G(e)$ has a recursive solution; let M_e be a machine that computes that solution. The set S and the indices of the machines M_e for all $e \in S$ are all finite information that can be incorporated into the following algorithm for a recursive solution of G. Given a number v first find out if $v \in \bigcup_{e\geq 0} A(e)$. If $v \notin \bigcup_{e\geq 0} A(e)$ then we need not match v, so output 0 and stop. If $v \in \bigcup_{e\geq 0} A(e)$ then we find out if $v \in \bigcup_{e\in S} A(e)$, then find e such that $v \in A(e)$, and then output $M_e(v)$. If $v \notin \bigcup_{e\in S} A(e)$ then run the construction until all the vertices in the same component as v are in the graph (this will happen since every component of $G(e)$ is finite). Since the component is a line graph with an even number of vertices, there is a unique solution on it. Output the vertex to which v is matched.

∎

6.4 Recursion-Theoretic Modification

Manaster and Rosenstein [118] showed that highly recursive bipartite graphs have solutions of low Turing degree.

Theorem 6.16 *If $G = (A, B, E)$ is a highly recursive bipartite graph that satisfies Hall's condition, then G has a solution M of low Turing degree.*

Proof:

Let $A = \{a_1 < a_2 < \cdots\}$. Consider the following recursive tree: The vertex $\sigma = (b_1, \ldots, b_n)$ is on T iff

1. for all i $(1 \leq i \leq n)$ we have $b_i \in nb_G(\{a_i\})$.

2. all the b_i's are distinct (so the vertex σ represents a solution of the first n vertices of A into B, namely a_i maps to b_i).

We have (1) T is recursive, (2) T is recursively bounded by the function $f(n) = \max_{1 \le i \le n} nb_G(\{a_i\})$, (3) any infinite branch of T is a solution of G, (4) every solution of G is represented by some infinite branch of T, and (5) the set of infinite branches of T is nonempty (by the classical Hall's theorem and the previous item). Since the branches of T form a nonempty Π^0_1 class, by Theorem 3.12 there exists an infinite low branch. This branch represents a solution of low degree. ∎

Theorem 6.17 *If $G = (A, B, E)$ is a recursive bipartite graph that satisfies Hall's condition, then G has a solution M such that $M' \le_{\mathrm{T}} \emptyset''$.*

Proof: If $G = (A, B, E)$, then the function nb is recursive in K. Define a tree T as in the previous theorem. Note that this tree is recursive in K and is K-recursively bounded. By Theorem 3.13 there exists an infinite branch B such that $B' \le_{\mathrm{T}} \emptyset''$. This branch represents the desired solution. ∎

6.5 Recursion-Combinatorial Modification

We now consider an effective version of Hall's theorem which is true. The modification is both recursion-theoretic and combinatorial.

Recall that by Theorem 6.5 a bipartite graph $G = (A, B, E)$ has a solution iff for all finite $X \subseteq A$, $|nb_G(X)| - |X| \ge 0$. A stronger condition would be to demand that $|nb_G(X)| - |X|$ is large for large $|X|$. In particular, if you want $|nb_G(X)| - |X| \ge n$ then there should be some n' such that $|X| \ge n'$ guarantees this. We formalize this:

Definition 6.18 A bipartite graph $G = (A, B, E)$ satisfies the *extended Hall's Condition* (e.H.c.) if there exists a function h such that $h(0) = 0$ and, for every finite $X \subseteq A$, $|X| \ge h(n) \Rightarrow |nb_G(X)| - |X| \ge n$. That is, to guarantee an 'expansion' of size n, take $|X| \ge h(n)$. Since $h(0) = 0$, e.H.c. implies Hall's condition.

Kierstead [96] proved the following effective version of Hall's theorem.

Theorem 6.19 *If $G = (A, B, E)$ is a highly recursive bipartite graph that satisfies e.H.c. with a recursive h, then G has a recursive solution. Moreover, given indices for G and h, one can effectively produce an index for a recursive solution.*

Proof:

Let a_0 be the first element of A. We plan to match a_0 with some b_0, define a function h', show that $G' = (A - \{a_0\}, B - \{b_0\}, E')$, where $E' = \{\{x, y\} : \{x, y\} \in E, x \in A - \{a_0\}, y \in B - \{b_0\}\}$, together with h', satisfies the hypothesis of the theorem, and iterate. This will easily produce a recursive solution.

Let

$$\begin{aligned} A_0 = &\ \{x \in A : \text{ there is a path from } x \text{ to } a_0 \text{ of length } \leq 2h(1)\}, \\ B_0 = &\ nb_G(A_0), \\ E_0 = &\ \{\{x, y\} \in E : x \in A_0, y \in B_0\}. \end{aligned}$$

Note that the vertices in B are of distance at most $2h(1) + 1$ from a_0.

Let G_0 be the finite bipartite graph (A_0, B_0, E_0). Clearly G_0 satisfies Hall's condition, so it has a solution. Let b_0 be the vertex to which a_0 is matched. Let h' be defined by (1) $h'(0) = 0$, (2) $(\forall n \geq 1) h'(n) = h(n + 1)$. Let G' be as indicated above. We show that G' satisfies e.H.c. via h'.

Let $n \in \mathsf{N}$, $X \subseteq A - \{a_0\}$, X finite, and $|X| \geq h'(n)$. We show that $|nb_{G'}(X)| - |X| \geq n$. We need only consider X such that $(X, nb_{G'}(X), E'')$ (where $E'' \subseteq E'$ is the set of edges between X and $nb_{G'}(X)$) is connected. There are several cases, depending on n and X.

Case 1: $n \geq 1$. Then $|X| \geq h'(n) = h(n+1)$. Hence $|nb_G(X)| - |X| \geq n + 1$. Hence $|nb_{G'}(X)| - |X| \geq n$.

Case 2: $n = 0$ and $b_0 \notin nb_G(X)$. Then $nb_{G'}(X) = nb_G(X)$. Hence $|nb_{G'}(X)| - |X| = |nb_G(X)| - |X| \geq 0$.

Case 3: $n = 0$ and $X \subseteq A_0$. Since (A_0, B_0, E_0) has a solution where a_0 maps to b_0, $|nb_{G'}(X)| - |X| \geq 0$.

Case 4: $n = 0$, $b_0 \in nb_G(X)$, and there exists $a \in X - A_0$ (this is negation of Cases 1,2, and 3). Since $b_0 \in nb_G(X)$ there exists a vertex $a' \in X$ such that $\{a', b_0\} \in E$. Since $(X, nb_{G'}(X), E'')$ is connected there exists a path $\langle a = x_0, x_1, \ldots, x_{2k} = a' \rangle$ in G'. Since G' is bipartite $x_i \in X$ iff i is even, hence there are at least k elements of X. Since $\{a', b_0\} \in E$ and $\{a_0, b_0\} \in E$ there is a path of length $2k + 1$ from a to a_0. The shortest path from a to

a_0 is of length $\geq 2h(1) + 1$, hence $2k + 1 \geq 2h(1) + 1$, so $k \geq h(1)$. Thus $|X| \geq h(1)$. ∎

Kierstead formulated the recursive e.H.c to prove the following corollary.

Corollary 6.20 *Let $G = (A, B, E)$ be a highly recursive bipartite graph. Assume there exists d such that (1) for all $x \in A$, $\deg(x) > d$, and (2) for all $x \in B$, $\deg(x) \leq d$. Then G has a recursive solution.*

Proof: We show that G together with the function $h(n) = dn$ satisfies the hypothesis of Theorem 6.19. Let X be a finite subset of A such that $|X| \geq dn$. We claim $|nb_G(X)| - |X| \geq n$. Consider the induced bipartite graph $(X, nb_G(X), E')$ where $E' = E \cap \{\{a, b\} \mid a \in X \wedge b \in nb_G(X)\}$. The number of edges is $\sum_{x \in X} \deg(x) \geq (d + 1)|X|$. But it is also $\sum_{y \in nb_G(X)} \deg(y) \leq d|nb_G(X)|$. Simple algebra yields $|nb_G(X)| - |X| \geq n$. ∎

Kierstead [96] has also shown that the assumption that h is recursive cannot be dropped.

6.6 Miscellaneous

We state several results about recursive solutions without proof.

Manaster and Rosenstein [118] investigated whether some other conditions on G yield recursive solutions. They showed that (1) if a highly recursive bipartite graph G has a finite number of solutions, then all those solutions are recursive , and (2) if a recursive bipartite graph G has a finite number of solutions, then all those solutions are recursive in K. However other conditions do not help: (1) there are highly recursive bipartite graphs where every vertex has degree 2 (this implies Hall's condition) which have no recursive solutions (this was extended to degree k in [119]), (2) there are decidable bipartite graphs (similar to decidable graphs, see Section 5.7.8) that satisfy Hall's condition but do not have recursive solutions.

McAloon [122] showed how to control the Turing degrees of solutions in graphs. He showed that there exists a recursive bipartite graph which satisfies Hall's condition and such that K is recursive in every solution. Along these lines, Manaster and Rosenstein (reported in [118]) showed that for every n, $1 \leq n \leq \aleph_0$, there exists a recursive bipartite graph with exactly n different solutions, and the n solutions are of n different Turing degrees. Manaster and Rosenstein also showed that for any Turing degree $\mathbf{a} \leq_{\mathrm{T}} \mathbf{0}'$ there exists a recursive bipartite graph that has a unique solution M, and M is of Turing degree $\mathbf{a}$. This yields a contrast to highly recursive graphs since any highly recursive bipartite graph with a unique solution has a recursive solution.

Manaster and Rosenstein [118] examined symmetric solutions in highly recursive bipartite graphs. A symmetric solution is a solution in $G = (A, B, E)$ which is a solution of B to A as well as A to B. The results are similar to those for solutions, and thus we do not state them.

Manaster and Rosenstein also showed that for any Turing degree $\mathbf{a}$ there exists a recursive bipartite graph that has a unique solution M, and M is of Turing degree $\mathbf{a}$.

Misercque [126] has refined the above theorems by showing the following: (1) given a (highly) recursive bipartite graph G, there exists a (recursively bounded) recursive tree T such that there is a bijection between the infinite paths through T and the solutions of G which preserves degree of unsolvability, (2) the analogue of (1) also holds for symmetric solutions, (3) for every (recursively bounded) recursive tree T there exists a (symmetric highly) recursive bipartite graph G such that there is a bijection between the infinite paths of T and the (symmetric) solutions of G, and (4) the analogue of (3) for arbitrary solutions is false (this disproved a conjecture of Jockusch and Soare from [90]).

Hirst [82] has proven several theorems about the proof-theoretic strength of Hall's Theorem. Several results in recursive solution theory can be derived as corollaries of his work.

7 Dilworth's Theorem for Partial Orders

We consider the infinite version of Dilworth's Theorem on partial orders. We (1) present the finite and infinite versions, which are both due to Dilworth [47, 48], (2) show that a recursive analogue of Dilworth's Theorem is false, (3) show that there is a recursion-theoretic modification that is true, (4) show that there is a combinatorial modification that is true, (5) show that there is a modification that is both recursion-theoretic and combinatorial which is true, and (6) state some miscellaneous results.

7.1 Definitions and Classical Version

Definition 7.1 A partial order $\mathcal{P}$ is a set P (called the *base set*) together with a relation $\leq$ that is transitive, reflexive, and anti-symmetric. The relation $<$ is defined by $x < y$ iff $x \leq y$ and $x \neq y$. If either $x \leq y$ or $y \leq x$, then x and y are *comparable*. If two elements x, y are not comparable, we denote this by $x|y$. A *chain* of $\mathcal{P} = \langle P, \leq \rangle$ is a set $C \subseteq P$ such that every pair of elements in C is comparable. A w-chain is a chain of size w. An *antichain* of $\langle P, \leq \rangle$ is a set $C \subseteq P$ such that every pair of elements in C is incomparable. A w-antichain is an antichain of size w. The *height* of $\langle P, \leq \rangle$ is the size of the largest chain. The *width* of $\langle P, \leq \rangle$, denoted $w(P)$, is the size of the largest antichain. If $w \in \mathsf{N}$, then a *w-cover* of $\langle P, \leq \rangle$ is a set of w disjoint chains such that every element of P is in some chain. We formally represent a w-cover as a function c from P to $\{1, \ldots, w\}$ such that if $c(x) = c(y)$ then x is comparable to y.

Dilworth's theorem states that if the largest antichain of a partial order is of size w, then it can be covered with w chains. The first published proof is by Dilworth [47, 48] is by induction on the width and is somewhat complicated [2] Other proofs have been given by Dantzig and Hoffman [44], Fulkerson [57], Gallai and Milgram [59], and Perles [134]. The most efficient algorithm to find a covering of a finite partial order $\langle P, \leq \rangle$ uses the computational equivalence of finding a maximum matching in a bipartite graph to finding a minimal covering (see [43] or [38, p. 339-341]) and runs in time $O(|P|^{2.5})$.

We present a simple proof of Dilworth's theorem due to Perles [134].

Theorem 7.2 (Finite Dilworth's Theorem) *If $\mathcal{P} = \langle P, \leq \rangle$ is a finite partial order of width w, then $\mathcal{P}$ has a w-cover.*

Proof:

We prove this by induction on $|P|$ (the size of P) for all w simultaneously. If $|P| = 1$ then the theorem is clear. Assume the theorem holds for all partial orders of size $\leq n - 1$. Let $\mathcal{P} = \langle P, \leq \rangle$ be a partial order such that $|P| = n$ and $w(\mathcal{P}) = w$. Let

$P_{\max} = \{x \in P : (\forall y \in P)[x \text{ comparable to } y \Rightarrow y \leq x]\}$,

[2] Erdos [51] claims that Galai and Milgram had a complete proof of this in 1947, and that Galai did not want this known in his lifetime since he was modest and did not want to seem like he was bickering about priority.

$P_{\min} = \{x \in P : (\forall y \in P)[x \text{ comparable to } y \Rightarrow y \geq x]\}$,

There are two cases.

Case 1: There exists a w-antichain $A = \{a_1, \ldots, a_w\}$ such that $A \neq P_{\max}$ and $A \neq P_{\min}$. Let $\mathcal{P}^+ = \langle P^+, \leq\rangle$ and $\mathcal{P}^- = \langle P^-, \leq\rangle$ where

$P^+ = \{x \in P : (\exists i)[a_i \leq x]\}$

$P^- = \{x \in P : (\exists i)[a_i \geq x]\}$

Clearly $P = P^+ \cup P^-$ (since $w(\mathcal{P}) = w$), $A = P^+ \cap P^-$ (since A is an antichain), $|P^+| < n$ (since $A \neq P_{\min}$), $|P^-| < n$ (since $A \neq P_{\max}$), and $w(\mathcal{P}^+) = w(\mathcal{P}^-) = w$. Apply the induction hypothesis to $\mathcal{P}^+$ and $\mathcal{P}^-$ to obtain w-coverings COV^+ of $\mathcal{P}^+$ and COV^- of $\mathcal{P}^-$. Assume, without loss of generality, that for all i, $1 \leq i \leq w$, $COV^+(a_i) = COV^-(a_i) = i$. Then $COV^+ \cup COV^-$ is a w-covering of $\mathcal{P}$.

Case 2: For all w-antichains A either $A = P_{\max}$ or $A = P_{\min}$. Let C be a chain that has one endpoint in $P_{\max}$ and one in $P_{\min}$ (such easily exists—take an element of $P_{\max}$ and work your way down). Note that C intersects every w-antichain of $\mathcal{P}$ (i.e., intersects $P_{\max}$ and $P_{\min}$), hence the width of $\mathcal{P}' = \langle P - C, \leq\rangle$ is $\leq w - 1$. Since $|P - C| < n$ we can apply the induction hypothesis to $\mathcal{P}'$ to yield a $(w-1)$-covering of $\mathcal{P}'$. This covering, together with C, yields a w-covering of $\mathcal{P}$. ∎

We now prove the infinite Dilworth's Theorem. We give a direct proof; it can also be proven by König's Lemma (Theorem 3.3).

Theorem 7.3 (Infinite Dilworth's Theorem) *If $\mathcal{P} = \langle P, \leq\rangle$ is a countable partial order of width w, then $\mathcal{P}$ has a w-cover.*

Proof: Assume, without loss of generality, that $P = \mathsf{N}$ (though of course $\leq$ need not have any relation to the ordering of N). Let $\mathcal{P}_s = \langle\{0, 1, \ldots, s\}, \leq\rangle$. Let c_s be a w-covering of $\mathcal{P}_s$ (such exists by Theorem 7.2). We use c_s to define c, a w-covering of $\mathcal{P}$. Let

$c(0) = \mu i[(\exists^\infty s) c_s(0) = i]$

$c(n+1) = \mu i[(\exists^\infty s) c_s(n+1) = i \wedge \bigwedge_{j=0}^{n} c_s(j) = c(j)]$

It is easy to see that c is a w-covering. ∎

The proof of Theorem 7.3 given above is noneffective. To see if the proof could have been made effective we will look at a potential analogue. In order to state this analogue we need some definitions.

Definition 7.4 A partial order $\langle P, \leq \rangle$ is *recursive* if both the set P and the relation $\leq$ are recursive.

We will represent recursive partial orders $\langle P, \leq \rangle$ by the Turing machines that determine their base set and relation. An index for a recursive partial order will be an ordered pair $\langle e_1, e_2 \rangle$, such that e_1 is an index for a Turing machine that decides P, and e_2 is an index for a Turing machine that decides $\leq$.

Definition 7.5 An index e is a *valid index for a recursive partial order* if $e = \langle e_1, e_2 \rangle$ and the following hold.

1. $e_1 \in TOT01$. Let P denote $\{x : \{e_1\}(x) = 1\}$.
2. $e_2 \in TOT01$.
3. The relation defined by $x \leq y$ iff $\{e_2\}(x, y) = 1$, when restricted to $P \times P$, is a partial order on P.

The *partial order represented by* e is $\langle P, \leq \rangle$. We denote this partial order $\mathcal{P}_e$. Note that if $\{e_2\}(x, y) = 0$ and $\{e_2\}(y, x) = 0$ then x and y are incomparable.

Definition 7.6 If $\mathcal{P}$ is a recursive partial order, then the *recursive width of* $\mathcal{P}$ is the least w such that $\mathcal{P}$ can be recursively covered with w recursive chains. (Theorems 7.2 and 7.3 justify this definition.) We denote the recursive width of $\mathcal{P}$ by $w^r(\mathcal{P})$.

Potential Analogue 7.7 There is a recursive algorithm $\mathcal{A}$ that performs the following. Given an index e for a recursive partial order $\mathcal{P} = \langle P, \leq^P \rangle$ of width w, $\mathcal{A}$ outputs an index for a recursive w-covering of $\mathcal{P}$. A consequence is that all recursive partial orderings $\mathcal{P}$ have $w(\mathcal{P}) = w^r(\mathcal{P})$.

Kierstead [94] showed that this Potential Analogue is false, but that a combinatorial modification is true. We have a recursion-theoretic modification which is true. Schmerl (reported in [94]) has a modification that is both recursion-theoretic and combinatorial which is true. In summary, the following are true:

1. For every $w \geq 2$, there exists a recursive partial order $\mathcal{P}$ such that $w(\mathcal{P}) = w$ but $w^r(\mathcal{P}) = \binom{w+1}{2}$ (proved by Szemérédi and Trotter, reported in [97]). For the case of $w = 2$ closer bounds are known: every recursive partial order of width 2 can be recursively 6-covered; however, there exists a recursive partial order of width 2 that cannot be recursively 4-covered.

2. There is a recursive algorithm $\mathcal{A}$ that performs the following. Given an index e for a recursive partial order $\mathcal{P} = \langle P, \leq^P \rangle$ of width w, $\mathcal{A}$ outputs an index for a recursive $\frac{5^w-1}{4}$-covering of $\mathcal{P}$ [94].

3. Every recursive partial order of width w has a w-covering that is low.

4. If $\mathcal{P}$ is a recursive locally finite partial order (defined in Section 7.6) then $w(\mathcal{P}) = w^r(\mathcal{P})$ (proven by Schmerl, reported in [94]). The proof does not yield an effective procedure to pass from indices of partial orders to indices of coverings.

We will prove subcases of i and iii to give the reader the ideas involved. The full proofs use the same recursion-theoretic ideas, but are more complicated combinatorially. Items ii and iv will be proven completely.

We will need the following definitions.

Definition 7.8 A partial order $\mathcal{Q} = \langle Q, \leq^Q \rangle$ *extends* a partial order $\mathcal{P} = \langle P, \leq^P \rangle$ if $P \subseteq Q$ and, for all $x, y \in P$, if $x \leq^P y$ then $x \leq^Q y$. (Note that elements that are incomparable in $\langle P, \leq^P \rangle$ might be comparable in $\langle Q, \leq^Q \rangle$.)

Definition 7.9 Let $\mathcal{P}_0, \mathcal{P}_1, \ldots$ be a (possibly finite) sequence of partial orders such that, for all i, $\mathcal{P}_{i+1}$ extends $\mathcal{P}_i$. Let $\mathcal{P}_j = \langle P_j, \leq_j \rangle$. Then *the union partial order of* $\mathcal{P}_0, \mathcal{P}_1, \ldots$ is $\langle \bigcup_{j=0}^{\infty} P_j, \leq \rangle$ where $x \leq y$ iff $(\exists j)[x, y \in P_j \wedge x \leq_j y]$. We denote this partial order by $\bigcup_j \mathcal{P}_j$.

7.2 Recursive Analogue is False

We show that there exists a recursive partial order of width w and recursive width $\binom{w+1}{2}$. Actually we prove something more general: for every u such that $w \leq u \leq \binom{w+1}{2}$ there is a recursive partial order $\mathcal{P}$ such that $w(\mathcal{P}) = w$

and $w^r(\mathcal{P}) = u$. The proof requires two lemmas; the first one is used in the second, and the second is similar in spirit to Lemma 5.13. The proof of the main theorem is similar in spirit to the proof of Theorem 5.15. The lemmas are more general then is needed for this section, but will be used again in Section 7.3.

Definition 7.10 Let $\{e\}$ be a Turing machine and let W be a set. If $(\forall x \in W)[\{e\}(x)\downarrow]$ then $\{e\}(W)$ is defined to be $\{\{e\}(x) : x \in W\}$.

In the following lemma we show that given a width $w \geq 1$ and a Turing machine $\{e\}$ we can construct a recursive partial order $\mathcal{P} = \langle P, \leq \rangle$ such that $w(\mathcal{P}) \leq w$, $w^r(\mathcal{P}) = w$, but $\{e\}$ will have a hard time covering $\mathcal{P}$. Formally either (1) there is an $x \in P$ such that $\{e\}(x)\uparrow$, (2) there are $x, y \in P$ that are incomparable and $\{e\}(x) = \{e\}(y)$, or (3) there is a chain $A = \{a_w < a_{w-1} < \cdots < a_1\}$ such that $|\{e\}(A)| = w$. If (1) or (2) occurs then $\{e\}$ is not a covering. If (3) happens then $\{e\}$ may still cover $\mathcal{P}$ but it has foolishly covered a single chain with w different chains.

Lemma 7.11 *Let $w \geq 1$, $\{e\}$ be a Turing machine, and X be an infinite recursive set. There exists a finite sequence of finite partial orders $\mathcal{P}_1, \ldots, \mathcal{P}_r$ such that $r \leq w$ and the following hold. (For notation $\mathcal{P}_j = \langle P_j, \leq_j \rangle$.)*

1. *For every j, $1 \leq j \leq r$, $\mathcal{P}_j$ is a tree with one branch of length j, denoted $A_j = \{a_j < \cdots < a_1\}$, and leaves consisting of $B_j \cup \{a_j\}$, where B_j is an anti-chain, $B_j = \{b_1, \ldots, b_k\}$ $(k \leq j-1)$, and $A_j \cap B_j = \emptyset$. Every element of B_j is placed directly below some element of A_j, but no element of B_j is above any element of A_j. Since any antichain contains at most one element from A_j, $w(\mathcal{P}_j) \leq k+1 \leq j \leq w$.*

2. *For every j, $1 \leq j \leq r-1$, (1) for all $x \in P_j$ $\{e\}(x)\downarrow$, (2) $\mathcal{P}_{j+1}$ can be obtained recursively from $\mathcal{P}_j$ (and the values of $\{e\}(x)$ for every $x \in P_j$), (3) $|\{e\}(A_j)| = |A_j| = j$, and $|\{e\}(B_j)| = |B_j|$. Also, $|\{e\}(B_r)| = |B_r|$.*

3. *For every j, $1 \leq j \leq r$, (1) $P_j \subseteq X$, and (2) canonical indices for the finite sets P_j and $\leq_j$ can be effectively obtained from e, j, w and an index for X. Note that r is not needed.*

4. *For every j, $2 \leq j \leq r$, (1) $\mathcal{P}_j$ is an extension of $\mathcal{P}_{j-1}$, (2) $A_{j-1} \subseteq A_j$, (3) $B_{j-1} \subseteq B_j$, (4) $a_j \in A_j - A_{j-1}$, (5) the only elements in $P_j - P_{j-1}$ are below and adjacent to a_{j-1}, and (6) $\{e\}(B_j) \subseteq \{e\}(A_{j-1})$.*

5. *At least one of the following occurs.*

   (a) *$\{e\}(a_r) \uparrow$ (so $\{e\}$ cannot recursively cover $\mathcal{P}_r$).*

   (b) *$(\exists x, y \in A_r \cup B_r)$ such that $x|y$ and $\{e\}(x) \downarrow = \{e\}(y) \downarrow$ (so $\{e\}$ cannot recursively cover $\mathcal{P}_r$).*

   (c) *$r = w$ and $\{e\}(a_w) \downarrow \notin \{e\}(A_{w-1})$. By b, $|\{e\}(A_{w-1})| = w - 1$, hence $|\{e\}(A_w)| = w$ (so if $\{e\}$ is trying to cover a partial order that includes $\mathcal{P}_r$, then it has just foolishly covered a single chain with w different chains).*

6. *$\mathcal{P}_r$ is a recursive partial order. Moreover, an index for both P_r and $\leq_r$ can be obtained from e, w and an index for X (note that we do not need r). The algorithm for P_r is as follows: given x, wait until x steps in the construction have gone by; if x has not entered the partial order by that step, it never will. The algorithm for $\leq_r$ is as follows: given x, y wait until $\max\{x, y\}$ steps in the construction have gone by; if x, y are both in the partial order then $x \leq y$ iff $x \leq y$ at that stage. (When an element enters the partial order, its relation to all the elements numerically less than it that are already in the partial order is known.)*

7. *The following algorithm produces a recursive covering of $\mathcal{P}_r$ that uses $|B_r| + 1 \leq w$ covers. Map a_1 to 1; whenever p enters the partial order map p to the least number that is not being used to cover some element incomparable to p (note that all elements already in the partial order will already be covered). This algorithm will map b_i to i. We will refer to this algorithm for covering as the greedy algorithm. It is easy to see that an index for the greedy algorithm can be effectively obtained from e, w and an index for X. (What needs to be proved is that the greedy algorithm yields a w-covering.)*

Proof:

The Turing machine $\{e\}$ is fixed throughout this proof.

We prove this lemma by induction on w. Assume $w = 1$ and x is the least element of X. Let $\mathcal{P}_1 = \mathcal{P}_r = (\{x\}, \emptyset)$. If $\{e\}(x) \uparrow$, then $v.a$ is satisfied.

If $\{e\}(x)\downarrow$, then $v.c$ is satisfied (vacuously). In either case conditions i–vii are easily seen to be satisfied. Note that $a_1 = x$.

Assume the lemma is true for w. We show it is true for $w+1$. Let $\mathcal{P}_1, \ldots, \mathcal{P}_r$ be the sequence of partial orders that exists via the induction hypothesis with parameter w.

If $v.a$ ($v.b$) holds for $\mathcal{P}_r$ with parameter w, then $v.a$ ($v.b$) holds for $\mathcal{P}_r$ with parameter $w+1$. Hence the sequence $\mathcal{P}_1, \ldots, \mathcal{P}_r$ is easily seen to satisfy i–vii.

If $v.c$ holds for $\mathcal{P}_r$, then note that $r = w$ and let $A_w = \{a_1, \ldots, a_w\}$ and $B_w = \{b_1, \ldots, b_k\}$ be as in condition i. Note that $\{e\}$ converges on all elements of A_w and $|\{e\}(A_w)| = w$. We construct an extension of $\mathcal{P}_r$. Initialize as follows.

1. Set p to be a new number chosen to be the least element of X that is bigger than both any element used so far, and the number of steps spent on this construction so far (this is done to make the partial order recursive).

2. Set Z to be B_w. Place p under a_w and incomparable to all elements in Z (we do not yet say if this new element is in A_{w+1} or B_{w+1}).

3. Set k' to be k.

Be aware that p, Z, and k' may change in the course of the construction. Note that all the elements in Z are pairwise incomparable. This will be true throughout the construction and easily provable inductively. Run $\{e\}(p)$. There are several cases; in all cases the only elements in $A_{w+1} - A_w$ or $B_{w+1} - B_w$ are those which we place there as follows.

1. $\{e\}(p)\downarrow \notin \{e\}(A_w)$. Set a_{w+1} to p, A_{w+1} to $A_w \cup \{a_{w+1}\}$, and B_{w+1} to Z. Since $|\{e\}(A_{w+1})| = |\{e\}(A_w)|+1 = w+1$, condition $v.c$ is satisfied.

2. $\{e\}(p)\downarrow \in \{e\}(Z) \cap \{e\}(A_w)$. Set a_{w+1} to p, A_{w+1} to $A_w \cup \{a_{w+1}\}$, and B_{w+1} to Z. Since p is incomparable to all elements in Z, condition $v.b$ is satisfied. (Setting a_{w+1} to p is only a technicality to make condition i true.)

3. $\{e\}(p)\downarrow \in \{e\}(A_w) - \{e\}(Z)$. Set k' to $k'+1$. Let $b_{k'}$ be p and place p into Z. Designate a new number to be p, chosen to be the least

element of X that is bigger than both any element used so far, and the number of steps spent on this construction so far (this is done to make the partial order recursive). Place p under a_w and incomparable to all elements of Z. Run $\{e\}(p)$ considering these cases $1, 2, 3$, and 4. Note that every time case 3 occurs, $|Z|$ grows by one and $|\{e\}(Z)| = |Z|$. Since $|A_w| = w$ (inductively using condition i), case 3 can occur at most $w - k$ times.

4. $\{e\}(p) \uparrow$. We (nonconstructively) set a_{w+1} to p, A_{w+1} to $A_w \cup \{a_{w+1}\}$, and B_{w+1} to Z. Condition $v.a$ is satisfied. (The $w + 2$ case is unaffected by this nonconstructiveness since $\{e\}(p)$ diverging yields a trivial induction step.)

In any case it is obvious how to define $\mathcal{P}_{r+1}$. It is easy to see that in any case conditions *i*–*vi* hold. We need to show that *vii* holds, namely that the greedy algorithm will $(w + 1)$-cover $\mathcal{P}_{r+1}$. By the induction hypothesis the greedy algorithm $(k + 1)$-covers $\mathcal{P}_r$ and covers b_i with i. Let $B_{w+1} = \{b_1, \ldots, b_k, b_{k+1}, \ldots, b_{k'}\}$. We prove, by induction on $i \geq k + 1$, that the greedy algorithm will cover b_i with i. Let $i \geq k + 1$. When b_i enters the partial order the greedy algorithm will cover it with i since $b_1, \ldots, b_{i-1}$ are covered with $\{1, \ldots, i - 1\}$ and are the only elements that are incomparable to b_i. Note that there are $k' \leq w$ elements of B_{r+1} and exactly one element in $A_{r+1} - A_r$ (namely a_{w+1}). The element a_{w+1} will be covered with $k'+1 \leq w+1$ when it enters. Hence the greedy algorithm provides a $(k' + 1)$-covering with b_i getting covered with i. ∎

In the following lemma we show that given a width $w \geq 1$, a number u such that $w \leq u \leq \binom{w+1}{2}$, and a Turing machine $\{e\}$ we can construct a recursive partial order $\mathcal{Q} = \langle Q, \leq \rangle$ such that $w(\mathcal{Q}) \leq w$, $w^r(\mathcal{Q}) \leq u$ and $\{e\}$ is not a $u - 1$-covering of $\mathcal{Q}$. Formally either (1) there is an $x \in Q$ such that $\{e\}(x) \uparrow$, (2) there are $x, y \in Q$ that are incomparable and $\{e\}(x) = \{e\}(y)$, or (3) there is a set $W \subseteq Q$ such that $|\{e\}(W)| = u$. If (1) or (2) occurs then $\{e\}$ is not a covering. If (3) happens then $\{e\}$ may still cover $\mathcal{Q}$ but it has to use at least u different chains.

Lemma 7.12 *Let $w \geq 1$, $\{e\}$ be a Turing machine, and X be an infinite recursive set. Let u be any number such that $w \leq u \leq \binom{w+1}{2}$. There exists a finite sequence of finite partial orders $\mathcal{Q}_1, \ldots, \mathcal{Q}_r$ such that the following hold. (For notation $\mathcal{Q}_j = \langle Q_j, \leq_j \rangle$.)*

1. *For every j, $2 \leq j \leq r$, $\mathcal{Q}_j$ is an extension of $\mathcal{Q}_{j-1}$. For each j, $1 \leq j \leq r$ (1) $Q_j \subseteq X$, and (2) canonical indices for the finite sets Q_j and $\leq_j$ can be effectively computed given e, j, u, w and an index for X.*

2. *For every j, $1 \leq j < r$, (1) for all $x \in Q_j$ $\{e\}(x) \downarrow$, (2) $\mathcal{Q}_{j+1}$ can be obtained recursively from $\mathcal{Q}_j$ (and the values of $\{e\}(x)$ for every $x \in Q_j$).*

3. *One of the following holds.*

   (a) *$(\exists x \in Q_r)[\{e\}(x) \uparrow]$ (so $\{e\}$ cannot be a cover of $\mathcal{Q}_r$).*

   (b) *$(\exists x, y \in Q_r)[\{e\}(x) \downarrow = \{e\}(y) \downarrow$ and $x|y$] (so $\{e\}$ cannot be a cover of $\mathcal{Q}_r$).*

   (c) *$(\exists W \subseteq Q_r)[|W| = |\{e\}(W)| = u]$ (so $\{e\}$ cannot be a $(u-1)$-covering of $\mathcal{Q}_r$).*

4. *$\mathcal{Q}_r$ is a recursive partial order. Moreover, an index for both Q_r and $\leq_r$ can be obtained from e, u, w and an index for X (note that we do not need r). The algorithm is similar to that in Lemma 7.11.vi.*

5. *$w(\mathcal{Q}_r) \leq w$.*

6. *$w^r(\mathcal{Q}_r) \leq u$. Moreover, an index for a u-covering of $\mathcal{Q}_r$ can be effectively obtained from e, u, w and an index for X.*

Proof:

The Turing machine $\{e\}$ is fixed throughout this proof.

We prove this lemma by induction on w. If $w = 1$, then let $\mathcal{Q}_1 = \mathcal{Q}_r = (\{x\}, \emptyset)$ where x is the least element of X. If $\{e\}(x) \uparrow$, then *iii.a* holds. If $\{e\}(x) \downarrow$, then *iii.c* holds with $W = \{x\}$. In either case conditions *i–vi* are easily seen to be satisfied.

Assume the lemma is true for w (and for all u with $w \leq u \leq \binom{w+1}{2}$). We show it for $w+1$ and any u such that $w+1 \leq u \leq \binom{w+2}{2}$. Recursively partition X into two infinite recursive sets Y and Z.

Let $\mathcal{P}_1, \ldots, \mathcal{P}_{r_1}$ be the sequence that exists via Lemma 7.11 with parameters $w+1, Y$. For $1 \leq j \leq r_1$ let $\mathcal{Q}_j = \mathcal{P}_j$. If *v.a* (*v.b*) of Lemma 7.11 holds for $\mathcal{P}_{r_1}$ then *iii.a* (*iii.b*) holds for $\mathcal{Q}_{r_1}$. Since conditions *i–v* (of this lemma) are easily seen to be satisfied, we are almost done (we did not use the induction hypothesis). We will later see why vi is true.

We now assume that $v.c$ of Lemma 7.11 holds for $\mathcal{P}_{r_1}$. Let A denote A_{r_1}, B denote B_{r_1}, and $A = \{a_{w+1} < \cdots < a_1\}$ ('$<$' is the order on P_{r_1}). Recall that for $1 \le j \le r_1$ we have set $\mathcal{Q}_j = \mathcal{P}_j$.

Note that $u \le \binom{w+2}{2} = \binom{w+1}{2} + w + 1$. Let i be the least number such that $u - i \le \binom{w+1}{2}$. Note that $0 \le i \le w+1$. It is easy to see that $w \le u - i \le \binom{w+1}{2}$ (this uses $w+1 \le u$). Apply the induction hypothesis with parameters w, $u-i$, and Z to obtain the sequence $\mathcal{Q}'_1, \ldots, \mathcal{Q}'_{r_2}$. If $i = 0$ then this sequence satisfies i–v and we are done. For the rest of the proof assume $i \ge 1$. We now construct $\mathcal{Q}_{r_1+1}, \ldots, \mathcal{Q}_{r_1+r_2}$. For j, $r_1 + 1 \le j \le r_1 + r_2$, set $\mathcal{Q}_j = \langle Q'_{j-r_1} \cup P_{r_1}, \le_j \rangle$ where $\le_j$ is defined as follows.

1. If $x, y \in Q'_{j-r_1}$ (P_{r_1}) then order as in Q'_{j-r_1} (P_{r_1}).
2. If $x \in \{a_1, \ldots, a_i\}$ and $y \in Q'_{j-r_1}$, then set $x|y$.
3. If $x \in \{a_{i+1}, \ldots, a_{w+1}\} \cup B$ and $y \in Q'_{j-r_1}$ then $x <_j y$.

(Note that we needed $1 \le i \le w+1$.) We examine the width of $\mathcal{Q}_{r_1+r_2}$. If C is an antichain of $\mathcal{Q}_{r_1+r_2}$, then one of the following occurs. (a) $C \cap B \ne \emptyset$, so $C \cap Q'_{r_2} = \emptyset$. Hence $C \subseteq P_{r_1}$ and since P_{r_1} has width at most $w+1$, $|C| \le w+1$, (b) $C \cap B = \emptyset$, so $C \cap P_{r_1} \subseteq A$. Since A is a chain, $|C \cap P_{r_1}| \le 1$. Since Q'_{r_2} has width at most w, $|C \cap Q_{r'_2}| \le w$. Hence $|C| \le w+1$. Since both cases lead to $|C| \le w+1$, $\mathcal{Q}_{r_1+r_2}$ has width $\le w+1$.

We now prove that condition iii holds for $\mathcal{Q}_{r_1+r_2}$. If $iii.a$ $(iii.b)$ holds for $\mathcal{Q}'_{r_2}$ then $iii.a$ $(iii.b)$ holds for $\mathcal{Q}_{r_1+r_2}$. If $iii.c$ holds for $\mathcal{Q}'_{r_2}$ then let $W' \subseteq Q'_{r_2}$ be such that $|\{e\}(W')| = u - i$. If $\{e\}(W') \cap \{e\}(\{a_1, \ldots, a_i\}) \ne \emptyset$, then there exist x, y such that $x|y$ and $\{e\}(x) = \{e\}(y)$, so $iii.b$ holds. If $\{e\}(W') \cap \{e\}(\{a_1, \ldots, a_i\}) = \emptyset$ then set W to be $W' \cup \{a_1, \ldots, a_i\}$. Note that $|\{e\}(W)| = |\{e\}(W')| + i = u - i + i = u$. Hence $iii.c$ holds.

By Lemma 7.11 we can effectively obtain, from e, w and an index for Y, an index for a $(w+1)$-covering COV_1 of $\mathcal{P}_{r_1}$. Inductively, we can effectively obtain, from $e, u-i, w$ and an index for Z, an index for a recursive $u-i$-covering COV_2 of $\mathcal{Q}'_{r_2}$. (Since we can obtain i from u, w we can also obtain the index for COV_2 from e, u, w and an index for Z.) Recall that, by the definition of a cover, COV_2 has range $\{1, \ldots, u-i\}$. We define a u-covering

COV of $\mathcal{Q}_{r_1+r_2}$ via

$$COV(x) = \begin{cases} COV_1(x) & \text{if } x \in P_{r_1}; \\ COV_2(x) + i & \text{if } x \in Q'_{r_2}. \end{cases}$$

To compute $COV(x)$ do the following. Given x, first find if $x \in Y$ or $x \in Z$ (if it is in neither then stop and output 1). If it is in Y (Z) then run the construction of the sequence of $\mathcal{P}_j$ (sequence of $\mathcal{Q}'_j$) until either x appears, or the number of steps used is larger than x (in which case x never will appear, so output 1). If x does appear then compute and output $COV_1(x)$ ($COV_2(x) + i$). Note that to construct an index for this function we only needed indices for COV_1 and COV_2, we did not need to know the manner in which the sequence of $\mathcal{P}_j$ or $\mathcal{Q}'_j$ succeeded in meeting its requirements. Hence we can effectively obtain this index even if the sequence of $\mathcal{P}$'s satisfies $v.c$ of Lemma 7.11.

It is clear that the range of COV is a subset of $\{1, \ldots, u\}$. We show that COV is a covering. If $COV(x) = COV(y)$ then either (1) $x, y \in P_{r_1}$ ($x, y \in Q'_{r_2}$), in which case x and y are comparable since $COV_1(x) = COV_1(y)$ ($COV_2(x) = COV_2(y)$), (2) $x \in P_{r_1} - \{a_1, \ldots a_i\}$ and $y \in Q'_{r_2}$ (or vice versa) in which case $x <_{r_1+r_2} y$ by definition of $<_{r_1+r_2}$. (The case $x \in \{a_1, \ldots, a_i\}$ and $y \in Q'_{r_2}$ cannot occur since then $COV(x) \in \{1, \ldots, i\}$ and $COV(y) \in \{i+1, \ldots, u\}$.) Hence COV is a u-cover. ∎

The following lemma is similar to Lemma 7.12 except that we make $w(\mathcal{Q}) = w$ instead of $w(\mathcal{Q}) \leq w$.

Lemma 7.13 *Let $w \geq 1$, $\{e\}$ be a Turing machine, and X be an infinite recursive set. Let u be any number such that $w \leq u \leq \binom{w+1}{2}$. There exists a finite sequence of finite partial orders $\mathcal{Q}_1, \ldots, \mathcal{Q}_r$ such that the following hold. (For notation $\mathcal{Q}_j = \langle Q_j, \leq_j \rangle$.)*

1. *There exists a w-antichain A such that, for all j, $A \subseteq Q_j$, and all elements of A are less than all elements of $Q_j - A$. For each j, $1 \leq j \leq r$, (1) $Q_j \subseteq X$, and (2) canonical indices for the finite sets Q_j and $\leq_j$ can be effectively computed given e, i, j, and an index for X.*

2. *For every j, $1 \leq j \leq r - 1$, Q_{j+1} can be effectively obtained from Q_j and the values of $\{e\}(x)$ for every $x \in Q_j - A$.*

3. *$\{e\}$ is not a $u-1$ covering of $\mathcal{Q}_r$.*

4. *$w(\mathcal{Q}_r) = w$ (this is the difference between this Lemma and Lemma 7.12).*

5. *$\mathcal{Q}_r$ is a recursive partial order. Moreover, an index for both Q_r and $\leq_r$ can be obtained from e, u, w and an index for X (note that we do not need r). The algorithm is similar to that in Lemma 7.11.vi.*

6. *$w^r(\mathcal{Q}_r) \leq u$. Moreover an index for a u-covering of $\mathcal{Q}_r$ can be effectively obtained from e, u, w and an index for X.*

Proof:

Let $A = \{x_1, \ldots, x_w\}$, the first w elements of X. Let $\mathcal{Q}'_1, \ldots, \mathcal{Q}'_r$ be the sequence obtained by applying Lemma 7.12 to parameters e, u, w, and $X - \{x_1, \ldots, x_w\}$. For notation $\mathcal{Q}'_j = \langle Q'_j, \leq'_j \rangle$. For all j, $1 \leq j \leq r$, let $\mathcal{Q}_j = \langle Q'_j \cup A, \leq_j \rangle$ where $\leq_j$ is defined by the following: (1) if $x, y \in A$ then $x|y$, (2) if $x, y \in Q'_j$ then $x \leq_j y$ iff $x \leq'_j y$, and (3) if $x \in A$ and $y \in Q'_j$ then $x \leq_j y$. ∎

In Lemma 7.13 we showed, given e, u, w, how to create a recursive partial order $\mathcal{P}$ such that $w(\mathcal{P}) = w$, $w^r(\mathcal{P}) \leq u$, and $\{e\}$ does not $u-1$-cover $\mathcal{P}$. We now combine all these partial orders to get a partial order that has width w, recursive width $\leq u$, and cannot be $u-1$-covered by any $\{e\}$. Hence its recursive width is exactly u.

Theorem 7.14 *Let $w \geq 2$ and u be such that $w \leq u \leq \binom{w+1}{2}$. Let X be an infinite recursive set. There exists a recursive partial order $\mathcal{P} = \langle P, \leq^P \rangle$ such that $w(\mathcal{P}) = w$, $w^r(\mathcal{P}) = u$, and $P \subseteq X$. (Note that if $w \in \{0, 1\}$ then for all recursive partial orders $\mathcal{P}$ such that $w(\mathcal{P}) = w$ we have $w^r(\mathcal{P}) = w(\mathcal{P})$.)*

Proof:

Let $X = \bigcup_{e \geq 0} X_e$ be a recursive partition of X into infinite sets. Let $\mathcal{Q}(e) = \langle Q_e, \leq_e \rangle$ be the partial order constructed in Lemma 7.13 using parameters e, u, w and X_e. Let $\mathcal{P} = \langle \bigcup_{e \geq 0} Q(e), \leq \rangle$ where $\leq$ is defined by (1) if $(\exists e)[x, y \in Q(e)]$ then $x \leq y$ iff $x \leq_e y$, (2) if $x \in Q(e_1)$ and $y \in Q(e_2)$ then $x \leq y$ iff x is bigger than y numerically. Clearly $\mathcal{Q}$ is a recursive partial order and $w(Q) = w$. Since for all e $\{e\}$ is not a $u-1$ covering of $\mathcal{Q}(e)$, $w^r(\mathcal{Q}) \geq u$. Since for all e $Q(e)$ is recursively u-coverable in a uniform way, $w^r(\mathcal{Q}) \leq u$. Combining the two inequalities yields $w^r(\mathcal{Q}) = u$. ∎

7.3 How Hard is it to Determine $w^r(\mathcal{P})$?

In this section we show that, even if $w(\mathcal{P})$ is known, and $w^r(\mathcal{P})$ is narrowed down to two prespecified values, it is Σ_3-complete to determine $w^r(\mathcal{P})$. By contrast the following promise problem is Π_1-complete: (D, A), where $D = \{e \mid e \text{ is the index of a recursive partial order}\}$ and $A = \{e \in D \mid \text{the partial order represented by } e \text{ has width} \leq w\}$.

The next lemma 'slows down' the construction of Lemma 7.12.

Lemma 7.15 *Let $w \geq 1$. Let $\{e\}$ be a a Turing machine, and X an infinite recursive set. Let u be any number such that $w \leq u \leq \binom{w+1}{2}$. There exists an infinite sequence of (not necessarily distinct) partial orders $\mathcal{R}_1, \mathcal{R}_2, \ldots$ such that the following hold. (For notation $\mathcal{R}_s = \langle R_s, \leq_s \rangle$.)*

1. *$\mathcal{R}_{s+1}$ is an extension of $\mathcal{R}_s$.*
2. *For all s, $R_s \subseteq X$ and $w(\mathcal{R}_s) = w$.*
3. *Given e, u, w and an index for X one can effectively find canonical indices for the finite sets R_s and $\leq_s$.*
4. *There exists a finite partial order $\mathcal{R} = \langle R, \leq \rangle$ and a number t such that $\mathcal{R} = \mathcal{R}_t$, and $(\forall s \geq t)\mathcal{R}_s = \mathcal{R}$. We call this partial order $\lim_{s\to\infty} \mathcal{R}_s$.*
5. *$\mathcal{R}$ is not $(u-1)$-covered by $\{e\}$.*
6. *$\mathcal{R}$ is a recursive partial order. Moreover, an index for both R and $\leq$ can be obtained from e, u, w and an index for X. The algorithm is similar to that in Lemma 7.11.vi.*
7. *Let x, y, s be such that $x, y \in R_s$ and s is the least such number. Then for all $t \geq s$, $x \leq_s y$ iff $x \leq_t y$, i.e., if elements are initially incomparable then they remain incomparable.*
8. *$w(\mathcal{R}) = w$.*
9. *$w^r(\mathcal{R}) \leq u$. Moreover, given e, u, w and an index for X one can effectively find an index for a recursive u-covering of $\mathcal{R}$.*

Proof:

Apply Lemma 7.13 to the parameters e, u, w, X. Break the construction of $\mathcal{Q}_r$ into stages such that at each stage either nothing is added to the partial order (e.g., one more step of the relevant Turing machine was run and did not converge), or an element is added and its relation with everything that is already in the partial order is established. Let $\mathcal{R}_s$ be the partial order produced at the end of stage s. It is easy to see that *i*–*ix* are satisfied. ▮

The next lemma is a 'parameterized version' of Lemma 7.13. Given e, u, w and a parameter y we construct a partial order $\mathcal{P}$ such that (1) $w(\mathcal{P}) = w$, (2) $w^r(\mathcal{P}) \le u$, and (3) if $y \notin TOT$ then $\mathcal{P}$ is not $u-1$-covered by $\{e\}$, and if $y \in TOT$ then $\mathcal{P}$ will be recursively w-covered (in this case we do not care about what $\{e\}$ does).

Lemma 7.16 *Let $w \ge 1$. Let $\{e\}$ be a a Turing machine, and X an infinite recursive set. Let u be any number such that $w \le u \le \binom{w+1}{2}$. Let $y \in \mathsf{N}$. There exists a recursive partial order $\mathcal{P} = (P, \le)$, which depends on y, such that the following hold.*

1. *$P \subseteq X$.*
2. *Given e, u, w, y and an index for X, one can effectively find an index for $\mathcal{P}$. The algorithm is similar to the one used in Lemma 7.11.vi.*
3. *$\mathcal{P}$ consists of a (possibly finite) set of finite partial orders $\mathcal{P}^1 = \langle P^1, \le^1 \rangle$, $\mathcal{P}^2 = \langle P^2, \le^2 \rangle$, $\mathcal{P}^3 = \langle P^3, \le^3 \rangle \ldots$ such that for all $i \ne j$ $P^i \cap P^j = \emptyset$ and all elements of P^j are less than (using the order $\le$ of $\mathcal{P}$) all elements of P^{j+1}. The $\mathcal{P}^j$'s are called constituents. The function that takes $\langle j, x \rangle$ and tells whether $x \in P^j$ is recursive.*
4. *If $y \notin TOT$ then*
   (a) *$\mathcal{P}$ consists of a finite number of constituents, and*
   (b) *$\mathcal{P}$ is not $(u-1)$-covered by $\{e\}$.*
5. *If $y \in TOT$ then*
   (a) *$\mathcal{P}$ consists of an infinite number of constituents, and*

(b) *given e, u, w, y and an index for X, and $p \in P$, one can effectively find the constituent containing p (i.e. find all the elements in the constituent and how they relate to p).*

(c) *$w^r(\mathcal{P}) = w$. This will follows from a,b, the finiteness of the constituents, and $w(\mathcal{P}) = w$ (the next item).*

6. *$w(\mathcal{P}) = w$.*

7. *$w^r(\mathcal{P}) \leq u$. Moreover, given e, u, w, y and an index for X one can effectively find an index for a recursive u-covering of $\mathcal{P}$.*

Proof:

We consider e, u, w, y and X fixed throughout this proof. Let $X = \bigcup_{j=0}^{\infty} X_j$ be a recursive partition of X into an infinite number of infinite recursive sets. Let $\mathcal{R}_1(j), \mathcal{R}_2(j) \ldots$ be the sequence of partial orders obtained by applying Lemma 7.15 to parameters e, u, w, X_j. (For notation $\mathcal{R}_s(j) = \langle R_s(j), \leq_{s,j} \rangle$.) We use these partial orders to construct $\mathcal{P}$ in stages. We denote the partial order at the end of stage s by $\mathcal{P}_s = \langle P_s, \leq_s \rangle$.

CONSTRUCTION

Stage 0: $\mathcal{P}_0 = \mathcal{R}_1(0)$, $j_0 = 0$, and $k_0 = 0$.

Stage s+1: Assume inductively that $\mathcal{P}_s = \langle \bigcup_{j=0}^{j_s} P_s^j, \leq_s \rangle$, $\mathcal{P}_s^{j_s} = \mathcal{R}_s(j_s)$, and that for all j, $0 \leq j \leq j_s - 1$, all the elements of $\mathcal{P}_s^j$ are $\leq_s$-less than all the elements of $\mathcal{P}_s^{j+1}$. Let k_{s+1} be the least element that is not in $W_{y,s}$. If $k_s = k_{s+1}$ then set $j_{s+1} = j_s$, else set $j_{s+1} = j_s + 1$. In either case set (1) for all $j < j_{s+1}$, $\mathcal{P}_{s+1}^j = \mathcal{P}_s^j$ and (2) $\mathcal{P}_{s+1}^{j_{s+1}} = \mathcal{R}_{s+1}(j_{s+1})$. (We refer to $\mathcal{R}_{s+1}(j_{s+1})$ as the *current partial suborder*.) Define $\leq_{s+1}$ as follows.

1. $x_1, x_2 \in \bigcup_{j=0}^{j_s} P_s^j$. So x_1, x_2 have been placed into the partial order in a previous stage. Set $x_1 \leq_{s+1} x_2$ iff $x_1 \leq_s x_2$.

2. $x_1, x_2 \in R_{s+1}(j_{s+1})$. Set $x_1 \leq_{s+1} x_2$ iff $x_1 \leq_{s+1, j_{s+1}} x_2$. (If $x_1, x_2 \in \bigcup_{j=0}^{j^s} P_s^j$ then this is not in conflict with case i. The relationship between x_1 and x_2 would have been set at a previous stage $s' \leq s$ via $x_1 \leq_{s'} x_2$ iff $x_1 \leq_{s', j_{s'}} x_2$ where $j_{s+1} = j_{s'}$. Note that by *vii* of Lemma 7.15 the relationship between x_1 and x_2 cannot change.)

3. $x_1 \notin R_{s+1}(j_{s+1})$ and $x_2 \in R_{s+1}(j_{s+1})$. So x_1 is not in the current partial suborder, but x_2 is. Set $x_1 \leq_{s+1} x_2$. (If $x_1, x_2 \in \bigcup_{j=0}^{j_s} P_s^j$ then this is

not in conflict with i since x_1 would have been set less than x_2 when x_2 enters the partial order, and via case iii.)

Note that if $k_{s+1} = k_s$ then we do not add any more constituents, we just add to the most recent one; and if $k_s \neq k_{s+1}$ then we create a new constituent and will never add to the previous constituents.

Set $\mathcal{P}_{s+1} = \langle \bigcup_{j=0}^{j_{s+1}} P_{s+1}^{j}, \leq_{s+1} \rangle$.

END OF CONSTRUCTION

Let $\mathcal{P} = \bigcup_s \mathcal{P}_s$. It is clear that $\mathcal{P}$ satisfies i and ii. Since for every j both $\mathcal{R}_s(j)$ and $\mathcal{R} = \lim_{s\to\infty} R_s(j)$ are finite, iii holds. By Lemma 7.15 each constituent of $\mathcal{P}$ is w-coverable therefore $\mathcal{P}$ has width w. Hence vi holds.

Assume $y \notin TOT$. Let k be the least element of $\overline{W}_y$. Let t be the least stage such that $0, 1, \ldots, k-1 \in W_{y,t}$. For all $s > t$, $j_s = j_{t+1}$; therefore $\mathcal{P}$ consists of a finite number of finite partial orders of the form $\mathcal{R}_{s'}(j')$ (where $s' < t$ and $j' < j_{t+1}$) along with $\mathcal{R} = \lim_{s\to\infty} \mathcal{R}_s(j_{t+1})$. Hence $iv.a$ holds. By Lemma 7.15, $\mathcal{R}$ is not $(u-1)$-covered by $\{e\}$, hence $iv.b$ holds.

Assume $y \in TOT$. Since $W_y = \mathsf{N}$, $\lim_{s\to\infty} j_s = \infty$. During every stage s such that $j_s \neq j_{s+1}$ a new constituent is created; therefore $\mathcal{P}$ consists of an infinite number of constituents. Hence $v.a$ holds.

To establish $v.b$ we show, given $p \in P$, how to find all the elements in the constituent containing p. Run the construction until $j, s \in \mathsf{N}$ are found such that p is an element of $R_s(j)$ (this will happen since $p \in P$). Run the construction further until t is found such that $j < j_t$ (this will happen since $y \in TOT$). The constituent of $\mathcal{P}_t$ that contains p is the constituent of $\mathcal{P}$ that contains p.

To establish $v.c$ we show that $w^r(\mathcal{P}) = w$. Given a number p, first test if $p \in P$. If $p \notin P$ then output 1 and halt (we need not cover it). If $p \in P$ then, using $v.b$, find all the elements of the constituent containing p. By vi this constituent has width w. Let c be the least lexicographical w-covering of this constituent. Output $c(p)$.

To establish vii we have to effectively find an index for a u-covering of $\mathcal{P}$ from e, u, w, y, and and index for X. Since $X = \bigcup_{j=0}^{\infty}$ is a recursive partition, we need only find, for each j, an index for the construction restricted to X_j, which we denote $\mathcal{P}[j]$. Let $\mathcal{R}[j]$ be the recursive partial order obtained by applying Lemma 7.15 with parameters e, u, w, X_j. Note that (1) $\mathcal{P}[j]$ is a suborder of $\mathcal{R}[j]$ and (2) we can effectively find an index e_j for a u-covering of $\mathcal{R}[j]$ from e, u, w, and an index for X_j. The index e_j restricted to the subset of X_j that is actually used, is an index for a u-covering of $\mathcal{P}[j]$. Note that this index is obtained without knowing if $y \in TOT$.

▮

Theorem 7.17 *Let $w \geq 2$. Let u be such that $w < u \leq \binom{w+1}{2}$. Let D be the set of all indices of recursive partial orders $\mathcal{P}$ such that $w(\mathcal{P}) = w$ and $w^r(\mathcal{P}) \in \{u, w\}$. Let $RWIDTH_{u,w}$ be the* 0-1 *valued partial function defined by*

$$RWIDTH_{u,w}(e) = \begin{cases} 1 & \text{if } e \in D \text{ and } w^r(\mathcal{P}_e) = w; \\ 0 & \text{if } e \in D \text{ and } w^r(\mathcal{P}_e) = u; \\ \text{undefined} & \text{if } e \notin D \ . \end{cases}$$

The promise problem $(D, RWIDTH_{u,w})$ is Σ_3-complete.

Proof: The following is a Σ_3 solution for $(D, RWIDTH_{u,w})$.

A_w is the set of ordered pairs $\langle e_1, e_2 \rangle \in TOT01$ such that there exists an i such that

1. $i \in TOT_w$, and
2. $(\forall x, y)[(\{i\}(x) = \{i\}(y) \wedge \{e_1\}(x) = \{e_1\}(y) = 1) \Rightarrow (\{e_2\}(x, y) = 1 \vee \{e_2\}(y, x) = 1)]$

(Recall that x, y are comparable iff either $\{e_2\}(x, y) = 1$ or $\{e_2\}(y, x) = 1$.)

We show that $(D, RWIDTH_{u,w})$ is Σ_3-hard by showing that if A is a solution to $(D, RWIDTH_{u,w})$ then $COF \leq_{\mathrm{m}} A$. Given x, we construct a recursive partial order $\mathcal{P}(x) = \mathcal{P}$ such that $w(\mathcal{P}) = w$ and

$x \in COF \Rightarrow w^r(\mathcal{P}) = w$, and

$x \notin COF \Rightarrow w^r(\mathcal{P}) = u$.

We use a modification of the construction in Theorem 7.14 of a recursive partial order which has width w but recursive width u. In this modification we weave the set W_x into the construction in such a way that if W_x is cofinite then the construction fails and $w^r(\mathcal{P}) = w$; and if W_x is not cofinite then the construction succeeds and $w^r(\mathcal{P}) = u$.

Let $\mathsf{N} = \bigcup_{e=0}^{\infty} X_e$ be a recursive partition of N into an infinite number of infinite recursive sets. Let y_e be defined such that

$$y_e \in TOT \text{ iff } \{e, e+1, \ldots\} \subseteq W_x$$

(it is easy to construct y_e from e). Let $\mathcal{P}(e) = \langle P(e), \leq_e \rangle$ be the recursive partial order obtained by applying Lemma 7.16 to e, u, w, X_e, y_e. Let $\mathcal{P} = \langle \bigcup_{e=0}^{\infty} P(e), \leq \rangle$ where $\leq$ is defined as follows: (1) if $x_1, x_2 \in P(e)$ then $x_1 \leq x_2$ iff $x_1 \leq_e x_2$, (2) if $x_1 \in P(e_1)$, $x_2 \in P(e_2)$ then $x_1 \leq x_2$ iff e_1 is numerically less than e_2. Clearly $\mathcal{P}$ is recursive and $w(\mathcal{P}) = w$.

If $x \notin COF$ then for all e we have $y_e \notin TOT$. Hence, by Lemma 7.16, for all e, $\mathcal{P}(e)$ is not $(u-1)$-covered by $\{e\}$. Therefore $w^r(\mathcal{P}) \geq u$. By Lemma 7.16, the partial orders $\mathcal{P}(e)$ are recursively u-coverable, and an index for a recursive u-covering can be obtained from e, u, w, y_e and an index for X_e. Hence $w^r(\mathcal{P}) \leq u$. Combining these two inequalities yields $w^r(\mathcal{P}) = u$.

If $x \in COF$ then $S = \{e \mid y_e \notin TOT\}$ is finite. Let $P' = \bigcup_{e \in S} P(e)$ and $P'' = \bigcup_{e \notin S} P(e)$. Let $\mathcal{P}' = \langle P', \leq' \rangle$ ($\mathcal{P}'' = \langle P'', \leq'' \rangle$) where $\leq'$ ($\leq''$) is the restriction of $\leq$ to P' (P''). We show that $\mathcal{P}$ is recursively w-coverable by showing that $\mathcal{P}'$ and $\mathcal{P}''$ are recursively w-coverable (and using that $P = P' \cup P''$ is a recursive partition of P).

If $e \in S$ then $y_e \notin TOT$ so, by Lemma 7.16, $\mathcal{P}(e)$ is finite and $w(\mathcal{P}(e)) = w$. Since S is finite, $\mathcal{P}'$ is a finite w-coverable partial order. Hence $w^r(\mathcal{P}') = w$. If $e \notin S$ then $y_e \in TOT$ so, by Lemma 7.16, $w^r(\mathcal{P}'') = w$. ∎

7.4 Combinatorial Modifications

Kierstead [94] proved that every recursive partial order of width w has recursive width $\leq \frac{5^w-1}{4}$. In Section 7.4.1 We present the $w = 2$ case of this theorem in detail, i.e., we show that every recursive partial order of width 2 can be recursively 6-covered. We then make remarks about how the proof for general w goes. We do not claim that from this one could reconstruct the proof for general w. In Section 7.4.2 we examine a modification where less information about the partial order is given; we provide no proofs.

By Theorem 7.14 there exist a recursive partial order of width 2 that cannot be covered by 3 recursive chains. Kierstead has shown that there exists a recursive partial order of width 2 that cannot be covered by 4 recursive chains

Also note that the lower bound of $\binom{w+1}{2}$ given in Theorem 7.14 cannot be tight since it fails for $w = 2$. The exact bound is unknown. It is open to find a w such that one can always recursively cover a partial order of width w with $\frac{\leq 5^w-1}{4}$ chains.

7.4.1 Bounding the Recursive Width

Notation 7.18 We often deal with several partial orders at the same time. In this case each partial order we deal with will have a superscript on the '$\leq$' symbol. Hence we use $\langle P, \leq^P \rangle$ for a partial order, $\leq^{\mathsf{N}}$ for the numerical order on the natural numbers, and $\leq^*$ for an order that we define. To indicate which order we are using, we use terms like 'N-greater than' or '$*$-comparable.'

Theorem 7.19 *If $\mathcal{P} = \langle P, \leq^P \rangle$ is a recursive partial order of width 2, then $\mathcal{P}$ has recursive width ≤ 6. Moreover, given an index for $\langle P, \leq^P \rangle$, one can recursively find an index for a recursive 6-covering of $\mathcal{P}$.*

Proof:

We define a recursive chain B and then show that $A = P - B$ can be recursively 5-covered. Let

$b_0 =$ the N-least element of P.

$b_{i+1} =$ the N-least x such that $b_i <^{\mathsf{N}} x$ and x is P-comparable to $b_1, \ldots, b_i$ (if no such x exists, then $b_{i+1} = b_i$).

$B = \{b_i \mid i \in \mathsf{N}\}$

The set B might be finite, but note that B is recursive and that an algorithm for it can be obtained effectively from an index for $\langle P, \leq^P \rangle$.

Note that B is a recursive chain and that for all $p \in A$ there exists $p' \in B$ such that $p' <^{\mathsf{N}} p$ and $p|p'$ (we will use this later).

By convention, elements of A will be denoted by small letters (e.g., p), elements of B will be denoted by small letters with primes (e.g. p'). Usually p and p' will be P-incomparable elements.

To show that A is recursively 5-coverable we will define a total ordering $\leq^*$ and an equivalence relation $\sim$ such that the following hold (the class that p is in is denoted $[p]$).

0a) Every equivalence class of $\sim$ is a $\leq^P$-chain.

0b) If $p <^* q <^* r$ and $p \sim r$, then $p \sim q \sim r$.

0c) If $x_1 \sim x_2$, $y_1 \sim y_2$, and $x_1 \not\sim y_1$, then $x_1 <^* y_1$ iff $x_2 <^* y_2$. Hence we can define $<^*$ and $\leq^*$ on equivalence classes via $[p] <^* [q]$ iff $[p] \neq [q]$ and $p <^* q$; $[p] \leq^* [q]$ iff $[p] = [q]$ or $[p] <^* [q]$. Both these definitions are independent of the representatives from $[p]$ or $[q]$ that are chosen.

$0d$) If $[p] <^* [q] <^* [r] <^* [s]$, then $p <^P s$.

$0e$) Both $\leq^*$ and $\sim$ are recursive.

We postpone the definitions of $\leq^*$ and $\sim$.

Notation 7.20 If $A \subseteq \mathsf{N}$, $n \in \mathsf{N}$, then A^n is the set containing the first n elements of A numerically.

Claim 0: If $\leq^*$ and $\sim$ can be defined to satisfy $0a, 0b, 0c, 0d$ and $0e$, then there is a recursive 5-covering of A.

Proof of Claim 0:

We describe a recursive 5-covering of A. Inductively assume that the elements of A^n have been distributed among 5 disjoint sets $C_1, \ldots, C_5$ such that

1. if $p \sim q$, then p and q are in the same C_i (hence we may speak of the set that $[p]$ is in),
2. if $p \not\sim s$ and $p, s \in C_i$, then there exist q and r such that either $[p] <^* [q] <^* [r] <^* [s]$ or $[s] <^* [r] <^* [q] <^* [p]$, and
3. each C_i is a $\leq^P$-chain (this follows from (2) and condition $0d$).

Let a be the numerically $(n+1)^{\text{st}}$ element of A. We determine which set to place a into by going through the following cases in order. (All elements referred to below, except a, are in A^n.)

Case 1: There exists $p \in C_i$ such that $p \sim a$. Then place a into C_i. It is easy to see that the inductive conditions still hold.

Case 2: There exists an i such that $C_i = \emptyset$. Let i_0 be the least such i. Place a into C_{i_0}. It is easy to see that the inductive conditions still hold.

Case 3: There exist r, s, t such that $[a] <^* [r] <^* [s] <^* [t]$, $t \in C_i$, and for no $t' \in C_i$ is $[t'] <^* [t]$. Then place a into set C_i. It is easy to see that the inductive conditions still hold.

Case 4: There exist o, p, q such that $[o] <^* [p] <^* [q] <^* [a]$, $o \in C_i$, and for no $o' \in C_i$ is $[o] <^* [o']$. Similar to Case 3.

Case 5: There exist elements o, p, q, r, s, t such that $[o] <^* [p] <^* [q] <^* [a] <^* [r] <^* [s] <^* [t]$, and there exists i such that $o, t \in C_i$, and for all u with

$[o] <^* [u] <^* [t]$, $u \notin C_i$. Place a into C_i. By 0d and the induction hypothesis, the inductive conditions still hold.

We show that at least one of these cases occurs. In particular, we assume that none of Cases 1,2,3, or 4 occur, and show that Case 5 holds. Let k (m) be the number of sets containing elements that are $*$-smaller ($*$-larger) than a. Let $C_{i_1}, \ldots, C_{i_k}$ ($C_{j_1}, \ldots, C_{j_m}$) be all the sets containing elements that are $*$-smaller ($*$-larger) than a. Let $b_{i_1}, \ldots, b_{i_k}$ ($b_{j_1}, \ldots, b_{j_m}$) be the $*$-largest ($*$-smallest) element of $C_{i_1}, \ldots, C_{i_k}$ ($C_{j_1}, \ldots, C_{j_m}$) that is $*$-smaller than a ($*$-larger than a). Without loss of generality assume the following holds:

$$[b_{i_k}] <^* \cdots <^* [b_{i_1}] <^* [a] <^* [b_{j_1}] <^* \cdots <^* [b_{j_m}].$$

We show that $k \geq 2$ (a similar proof shows $m \geq 2$). Assume, by way of contradiction, that $k \leq 1$. Since Case 2 does not hold, $m \geq 4$. Hence there exist three elements $r, s, t \in \{b_{j_1}, b_{j_2}, b_{j_3}, b_{j_4}\}$ such that $r <^* s <^* t$ and, if $k = 1$, $r, s, t \notin [b_{i_1}]$. Case 3 holds with these values of r, s, t, which is a contradiction.

Let $p = b_{i_2}, q = b_{i_1}, r = b_{j_1}, s = b_{j_2}$. Let o be some element that has been placed into a set, but $o \notin C_{i_1} \cup C_{i_2} \cup C_{j_1} \cup C_{j_2}$ (such exists since by the negation of Case 2 all five sets are used). Assume $o <^* a$ (the case $a <^* o$ is similar, though there we would call the element t instead of o). Since $o \notin [p] \cup [q]$ we have $[o] <^* [p] <^* [q] <^* [a]$. Let C be the set that o is in. We can assume that o is the $*$-largest such element of C that is $<^* a$. C must also contain some element t, $a <^* t$, else Case 4 holds. Let t be the $*$-least such element. Since $t \in C$, $t \notin [r] \cup [s]$. Hence $[a] <^* [r] <^* [s] <^* [t]$. Since o is the $*$-largest element of C that is $\leq^* a$, and t is the $*$-smallest element of C that is $\geq^* a$, the elements o, p, q, r, s, t satisfy Case 5.

End of Proof of Claim 0

We describe a relation $\leq^*$ on A and then prove that it is a recursive linear ordering.

Definition 7.21 If $p \in A$ then $inc(p) = \{p' : p' \in B \text{ and } p'|p\}$. If $p, q \in A$ then $inc(p) <^P inc(q)$ means that $\forall p' \in inc(p)\ \forall q' \in inc(q)\ p' <^P q'$. Let $inc(p) \leq^P inc(q)$ mean that either $inc(p) <^P inc(q)$ or $inc(p) = inc(q)$. If $inc(p) \leq^P inc(q)$ or $inc(q) \leq^P inc(p)$ then $inc(p)$ and $inc(q)$ are *P-comparable.*

Definition 7.22 We define a relation $\leq^*$ on A. We later show that $\leq^*$ is a recursive linear ordering. Given $p, q \in A$ apply the least case below that is satisfied by p and q.

1. if $p \leq^P q$ then $p \leq^* q$,
2. if $q \leq^P p$ then $q \leq^* p$,
3. if $inc(p) <^P inc(q)$ then $p \leq^* q$,
4. if $inc(q) <^P inc(p)$ then $q \leq^* p$.
5. if none of the above cases apply then p, q are $*$-incomparable (we later show this never occurs).

Claim 1:

1a) For every $p, q \in A$ either $p \leq^* q$ or $q \leq^* p$.

1b) $\leq^*$ is reflexive and transitive.

1c) $\leq^*$ is a recursive linear ordering.

1d) If $p, q, r \in A$, $p \leq^* q \leq^* r$, and $s' \in inc(p) \cap inc(r)$ then $s' \in inc(q)$.

1e) If $p <^* q$, $inc(p) \cap inc(q) \neq \emptyset$, then $p <^P q$.

Proof of Claim 1:
(1a): We show that if $p|q$ then either $inc(p) <^P inc(q)$ or $inc(q) <^P inc(p)$ (which implies that p and q are $*$-comparable). If not then there exist $p', p'' \in inc(p)$ and $q' \in inc(q)$ such that $p' <^P q' <^P p''$ (or the analogue with q', p', q''). Since P has width 2 and $p|q$, $q'|q$, we have that p is comparable to q'. However, $p <^P q'$ yields $p <^P p''$, and $q' <^P p$ yields $p' <^P p$, both of which contradict $p', p'' \in inc(p)$.
(1b): $\leq^*$ is clearly reflexive. We show that $\leq^*$ is transitive. Assume $p \leq^* q$ and $q \leq^* r$ and that p, q, and r are distinct. There are several cases to consider. (i) If $p \leq^P q$ and $q \leq^P r$ then $p \leq^P r$, hence $p \leq^* r$. (ii) If $inc(p) <^P inc(q)$ and $inc(q) <^P inc(r)$ then $inc(p) <^P inc(r)$, hence $p \leq^* r$. (iii) Assume $p \leq^P q$ and $inc(q) \leq^P inc(r)$. By 1a either $p \leq^* r$ or $r \leq^* p$. Assume, by way of contradiction, that $r \leq^* p$. If $r \leq^P p$ then $r \leq^P p \leq^P q$,

contradicting $q \leq^* r$. If $inc(r) <^P inc(p)$ then $inc(q) <^P inc(p)$. Hence we have $p <^P q$, $q' <^P p'$ (for all $p' \in inc(p)$ and $q' \in inc(q)$). The elements p and q' must be P-comparable since otherwise $q' \in inc(p)$ which violates $inc(q) <^P inc(p)$. But $p \leq^P q'$ implies $p \leq^P p'$, and $q' \leq^P p$ implies $q' \leq^P q$, both of which are contradictions. Hence $p \leq^* r$. (iv) Assume $inc(p) < inc(q)$ and $q \leq^P r$. Similar to iii.

$(1c)$: From $1a$ and $1b$ $\leq^*$ is a linear ordering. We describe an algorithm that determines how p and q $*$-compare. First determine how p and q P-compare. If $p \leq^P q$ then $p \leq^* q$, and if $q \leq^P p$ then $q \leq^* p$. If $p|q$, then by $1a$ $inc(p)$ and $inc(q)$ P-compare. Find $p' \in inc(p)$ and $q' \in inc(q)$. Now $p <^* q$ iff $p' <^P q'$.

$(1d)$: Assume by way of contradiction that $s' <^P q$. Then $q \not\leq^P r$ (else $s' <^P r$). Since $q <^* r$ and $q \not\leq^P r$, we have $inc(q) <^P inc(r)$. Since $s' \in inc(r)$, we have $inc(q) <^P s' <^P q$, which is a contradiction. Hence $s' \not\leq^P q$. Similar reasoning yields $q \not\leq^P s'$, so $s' \in inc(q)$.

$(1e)$: Let $x \in inc(p) \cap inc(q)$. Since P has width 2 and $p|x$, $q|x$ we know p and q are P-comparable. If $q <^P p$ then $q \leq^* p$, hence $p <^P q$.

End of Proof of Claim 1

We describe a recursive equivalence relation $\sim$ on A inductively. Assume that the elements of A^n have been put into equivalence classes. Given q, the numerically $(n+1)^{\text{st}}$ element of A, we proceed as follows. Find $q^-, q^+ \in A^n$ (if they exist) such that

q^- is the $*$-max element such that $q^- <^* q$.
q^+ is the $*$-min element such that $q <^* q^+$.
(Note that $q^- <^* q <^* q^+$.)

If there exists $q' \in B$, $q' <^{\mathsf{N}} q$, $q'|q$ and $q'|q^-$, then place q in the same class as q^-. If not, but if there exists $q' <^{\mathsf{N}} q$, $q'|q$ and $q'|q^+$ then place q in the same class as q^+. If neither of these occurs, then q becomes the first element of a new class.

We can now prove $0a, 0b$, and $0c$, the first three properties that were required of $\leq^*$ and $\sim$. We restate them because we need a slightly stronger version (strengthening the induction hypothesis).

Claim 2: For every n, when only the elements of A^n are put into classes, the following hold.

$2a$) If p, q are two $*$-adjacent elements such that $[p] = [q]$ then (1) $\exists x \in$

$inc(p) \cap inc(q)$ with $x <^N$ N-max$\{p, q\}$ and (2) p and q are P-comparable. Every class is a $\leq^P$-chain.

2b) If $p <^* q <^* r$ and $p \sim r$ then $p \sim q \sim r$.

2c) If $p_1 \sim p_2$, $q_1 \sim q_2$, and $p_1 \not\sim q_1$, then $p_1 <^* q_1$ iff $p_2 <^* q_2$.

Proof of Claim 2

By induction on n. Assume all three items are true for A^n and consider what may happen when q, the numerically $(n+1)^{\text{st}}$ element of A, is considered. Let q^-, q^+ be as in the definition of $\sim$. Note that q^- and q^+ are $*$-adjacent elements in A^n.

Case 1: If $q^- \sim q^+$ then q^- and q^+ are $*$-adjacent elements and $[q^-] = [q^+]$. By the induction hypothesis there exists $x \in inc(q^-) \cap inc(q^+)$ with $x <^{\mathsf{N}}$ N-max$\{q^-, q^+\}$. By Claim 1d, $x \in inc(q)$. Since $x <^{\mathsf{N}}$ N-max$\{q^-, q^+\} <^{\mathsf{N}} q$, the element q is placed in $[q^-]$. By Claim 1e, $q^- \leq^P q \leq^P q^+$. Hence 2a holds. It is easy to see that $2b, 2c$ hold as well.

Case 2: $q^- \not\sim q^+$ and $q^- \sim q$. Since q was placed in $[q^-]$, $(\exists x <^{\mathsf{N}} q)[x \in inc(q) \cap inc(q^-)]$. By Claim 1$e$, $q^- \leq^P q$. Hence, since $q = \mathsf{N}$-max$\{q, q^-\}$, 2a holds. It is easy to see that $2b, 2c$ hold as well.

Case 3: $q^- \not\sim q^+$ and $q^+ \sim q$. Similar to Case 2.

Case 4: q becomes the first element of a new class. In this case $2a, 2b$, and 2c hold trivially.

End of proof of Claim 2

Definition 7.23 We define $<^*$ and $\leq^*$ on classes via $[p] <^* [q]$ iff $[p] \neq [q]$ and $p <^* q$; and $[p] \leq^* [q]$ iff $[p] = [q]$ or $[p] <^* [q]$. Claim 2c shows that these definitions are independent of representation.

We need one more claim before we can prove item 0d about $\sim$.

Claim 3: Let a be the N-least element of $[a]$. Let b be such that $a <^* b$.

1. If there exists c' such that $c'|a$, $c'|b$, $c' <^{\mathsf{N}} a$ and $c' \in B$, then $a \sim b$.

2. If $a \not\sim b$ then for all $c' <^{\mathsf{N}} a$ such that $c' \in B$ and $c'|a$ we have $c' <^P b$.

3. If $a \not\sim b$ then there exists d such that $a \not\sim d$, $[a] <^* [d] \leq^* [b]$, d is the N-least element of $[d]$, and for all c' such that $c' <^{\mathsf{N}} d$, $c' \in B$, and $c'|d$, we have $a <^P c'$.

Proof of Claim 3:
i. We first show that $a <^{\mathsf{N}} b$. Let $S = \{d \in A : d <^{\mathsf{N}} a, a <^* d \leq^* b\}$. We show $S = \emptyset$ which easily implies $a <^{\mathsf{N}} b$. Assume, by way of contradiction, that $S \neq \emptyset$. Let d be the $*$-smallest element of S. Note that (1) since $a <^* d \leq^* b$, $c'|a$, and $c'|b$, by Claim 1d, $c'|d$, and (2) when a is placed into an equivalence class, d is the value of a^+ (i.e., the $*$-least element that is $*$-larger than a and N-less than a). Consider what happens when a is placed into an equivalence class. If a is placed into $[a^-]$ then a will not be the N-least element of $[a]$. If a is not placed into $[a^-]$ then, since $c'|a$, $c'|d$, $c' <^{\mathsf{N}} a$, and $d = a^+$, a is placed into $[d]$. In either case a is not the N-least element of its class, contrary to hypothesis. Hence $S = \emptyset$.

We now know that when b is placed into a class a has already been so placed. We show $b \sim a$ by induction on n, the number of elements N-smaller than b and $*$-between a and b when b is considered. If $n = 0$, then when b is considered $a <^* b$ (adjacent), $c'|a$, $c'|b$, and $c' <^{\mathsf{N}} a <^{\mathsf{N}} b$. Hence, b will be placed in $[a]$. If $n > 0$, then let b^- be the $*$-largest element that is $*$-less than b when b is placed. (so $a <^* \cdots <^* b^- <^* b$). Since $c'|a$ and $c'|b$, by Claim 1d $c'|b^-$. Since there are $\leq^{\mathsf{N}} n-1$ elements $*$-between a and b^- when b^- is placed, and since $c' <^{\mathsf{N}} a <^{\mathsf{N}} b^-$ (by the above argument) we may apply the induction hypothesis to a and b^-, hence $a \sim b^-$. Thus b will be placed into $[b^-] = [a]$, so $a \sim b$.
ii. By the contrapositive of i. we know that c' is comparable to b. Since $a <^* b$, either $a <^P b$ or $inc(a) <^P inc(b)$. Either case leads to $b \not\leq^P c'$, hence $c' <^P b$.
iii. If $b <^{\mathsf{N}} a$, let d be the N-least element of $[b]$ (we prove later that this choice of d works). If $a <^{\mathsf{N}} b$, let $S = \{d \in A : d \leq^{\mathsf{N}} b, a <^* d \leq^* b\}$. Let d^* be the $*$-least element of S such that $a \not\sim d^*$. (Note that d^* exists, since $b \in S$ and $a \not\sim b$.). Let d be the N-least element of $[d^*]$.

The remainder of this proof is valid for either choice of d. Since $a \sim a$, $d \sim d^*$, $a \not\sim d^*$, and $a <^* d$, by Claim 2c $[a] <^* [d]$. Since $d \leq^* b$, by definition $[d] \leq^* [b]$. Hence $[a] <^* [d] \leq^* [b]$.

Let $c'|d$, $c' <^{\mathsf{N}} d$, and $c' \in B$. We first show that c' and a are P-comparable, and second that $a <^P c'$. There are two cases.
Case 1: $c' <^{\mathsf{N}} a$. Note that $c'|d$, $c' <^{\mathsf{N}} a$, and $c' \in B$. If $c'|a$ then, by i (of this claim), $a \sim d$. Hence c' and a are P-comparable.
Case 2: $a <^{\mathsf{N}} c' <^{\mathsf{N}} d$. Note that $a <^{\mathsf{N}} d$. Hence when d is placed into a class a has already been so placed. Let d^- be as in the definition of classes. Note

that, since d is the N-least element of $[d]$, we have $[a] \leq^* [d^-] <^* [d]$. Note that $c'|d$. If $c'|a$ then, by Claim 1d, $c'|d^-$. But then d will be placed into $[d^-]$, which is a contradiction.

We show that $a <^P c'$. Since $a <^* d$ either $a \leq^P d$ or $inc(a) <^P inc(d)$. Either case implies $c' \not\leq^P a$. Since a, c' are P-comparable we have $a <^P c'$.
End of Proof of Claim 3

We can now prove 0d.
Claim 4: If $[p] <^* [q] <^* [r] <^* [s]$ then $p <^P s$.
Proof of Claim 4: We can take q to be the N-least element of $[q]$. Let d be as in Claim 3.iii with $a = q$, $b = r$, and let $q', d' \in B$ be such that $q|q'$, $q' <^{\mathsf{N}} q$, $d|d'$, and $d' <^{\mathsf{N}} d$. Then $[p] <^* [q] <^* [d] \leq^* [r] <^* [s]$, d is the N-least element of $[d]$, and $q <^P d'$. By Claim 3.ii with $a = d$, $b = s$, $c' = d'$ we obtain $d' <^P s$. Since $q <^P d'$, and q', d' are comparable (as all elements of B are), $q' <^P d'$ (else $q <^P d' \leq^P q'$). If $p <^P q$ then we have $p <^P q <^P d' <^P s$ so we are done. If $p|q$ then since $q|q'$ and $\langle P, \leq^P \rangle$ has width 2, p is comparable to q'. Assume, by way of contradiction, that $q' <^P p$. Since $p <^* q$ either (1) $p <^P q$ so $q' <^P p <^P q$, or (2) $inc(p) <^* inc(q)$ so $p' <^P q' <^P p$ for $p' \in inc(p)$. Hence we have $p <^P q'$, so $p <^P q' <^P d' <^P s$.
End of Proof of Claim 4 ▮

Theorem 7.24 *If $\langle P, \leq^P \rangle$ is a recursive partial order of width w then P has recursive width $\leq \frac{5^w - 1}{4}$. Moreover, given an index for $\langle P, \leq^P \rangle$ one can recursively find an index for a recursive $\frac{5^w-1}{4}$-covering.*

Proof sketch: This is a proof by induction. The base case of $w = 2$ is Theorem 7.19. Let $\langle P, \leq^P \rangle$ be a partial order of width $w \geq 3$. First, a linear suborder is defined similar to B in Theorem 7.19. Second, a recursive partial order $\leq^*$ on $A = P - B$ is defined which is somewhat similar to the $\leq^*$ in Theorem 7.19. Third, prove that $\langle A, \leq^* \rangle$ is a recursive partial order of width $w - 1$. By induction $\langle A \leq^* \rangle$ has recursive width $\leq \frac{5^{w-1}-1}{4}$. We then show that every recursive $*$-chain of A can be covered by 5 P-chains. Thus P can be recursively covered by $1 + 5(\frac{5^{w-1}-1}{4}) = \frac{5^w-1}{4}$ recursive chains. ▮

7.4.2 Bounding the Recursive Width Given Partial Information

In the algorithms in Theorems 7.19 and 7.24 we needed the ability to tell *how* elements compared. What happens if we only have the ability to tell *if* elements compare?

Definition 7.25 If $\mathcal{P} = \langle P, \leq \rangle$ is a partial order then the *co-comparability graph of* $\mathcal{P}$ is $G_{\mathcal{P}} = (P, E)$ where $E = \{\{x, y\} : x, y \text{ are incomparable}\}$. The set of all co-comparability graphs of finite or countable partial orders is denoted Γ_{co}.

Notation 7.26 If G is a graph then $\omega(G)$ is the size of the largest clique in G.

A (recursive) a-coloring of $G_{\mathcal{P}}$ yields a (recursive) a-covering of $\mathcal{P}$; and $w(\mathcal{P}) = \omega(G_{\mathcal{P}})$. Hence we examine recursive colorings of recursive graphs in $G \in \Gamma_{co}$ with fixed $\omega(G)$.

Schmerl asked if there exists a function f such that, for every recursive $G \in \Gamma_{co}$, $\chi^r(G) \leq f(\omega(G))$. Kierstead, Penrice, and Trotter [101] answered this affirmatively. Their result is a corollary of a theorem in combinatorics and is part of a fascinating line of research. We sketch that line of research and their theorem. For a fuller account of this area see [98].

Definition 7.27 A class of graphs $\mathcal{G}$ is χ-*bounded* if there exists a function f such that, for all $G \in \mathcal{G}$, $\chi(G) \leq f(\omega(G))$. (The function f need not be computable.)

Notation 7.28 If H is a graph then $Forb(H)$ is the set of graphs that do not contain H as an induced subgraph. ('$Forb$' stands for 'Forbidden'.)

The question arises as to which classes of graphs are χ-bounded. If $Forb(H)$ is χ-bounded then, by a result of Erdos and Hajnal [52], H is acyclic. Gyárfás [70] and Sumner [162] conjectured the converse, i.e., if T is a tree then the class Forb(T) is χ-bounded. Gyárfás [71] showed that if P is a path (i.e., a graph with $V = \{v_1, \ldots, v_k\}$ and $E = \{\{v_i, v_{i+1}\} : i = 1, \ldots, k-1\}$) then $Forb(P)$ is χ-bounded. Kierstead and Penrice [100] showed that if T is a tree of radius 2 (i.e., there is a vertex v such that for all vertices x there is a path of length ≤ 2 from v to x) then $Forb(T)$ is χ-bounded. More is known if we restrict attention to on-line colorings.

Definition 7.29 An *on-line graph* is a structure $G^< = (V, E, <)$ where $G = (V, E)$ is a graph and $<$ is a linear ordering of V (if V is infinite then $<$ has the order type of the natural numbers). $G^<$ is an *on-line presentation of* G.

Definition 7.30 Let $G^< = (V, E, <)$ and $V = \{v_1 < v_2 < \cdots\}$. An *on-line algorithm to color* $G^<$ is an algorithm that colors $v_1, v_2, \ldots$ in order so that the color assigned to v_i depends only on G restricted to $\{v_1, \ldots, v_i\}$.

Definition 7.31 A class of graphs $\mathcal{G}$ is *on-line* χ*-bounded* if there exists a function f such that for all $G \in \mathcal{G}$, for all on-line presentations $G^<$ of G, there exists an on-line algorithm to color $G^<$ with $\leq f(\omega(G))$ colors.

The question arises as to which classes of graphs are on-line χ-bounded. Chvátal [39] showed that $Forb(P_4)$ is on-line χ-bounded where P_n is the path on n vertices. Gyárfás and Lehel [72] showed that $Forb(P_5)$ is on-line χ-bounded, but that $Forb(P_6)$ is not. Hence if $Forb(T)$ is on-line χ-bounded for some tree T then T has radius 2. Kierstead, Penrice, and Trotter [101] proved this condition is not only necessary by also sufficient. Combining the results we have mentioned of Erdos-Hajnal, Gyárfás-Lehlel (that $Forb(P_6)$ is not χ-bounded), and Kierstead-Penrice-Trotter, one obtains the following.

Theorem 7.32 *Let G be a connected graph. $Forb(G)$ is on-line χ-bounded iff G is a tree of radius 2.*

Theorem 7.32 can be applied to Γ_{co}.

Corollary 7.33 *Γ_{co} is on-line χ-bounded.*

Proof: Let T be the tree formed by subdividing each edge of $K_{1,3}$ (i.e., place a new vertex on every edge of $K_{1,3}$). By a case analysis one can verify that $\Gamma_{co} \subseteq Forb(T)$. Since T has radius 2, by Theorem 7.32, $Forb(T)$ is on-line χ-bounded. ∎

Every recursive $G \in \Gamma_{co}$ has a recursive on-line presentation. Hence the on-line coloring algorithm in Corollary 7.33 yields a recursive coloring of all recursive graphs in Γ_{co}. Hence we have the following corollaries

Corollary 7.34 *There exists a function f such that*

1. *for every recursive $G \in \Gamma_{co}$, $\chi^r(G) \le f(\omega(G))$, and*

2. *for every partial order $\mathcal{P}$, $w^r(\mathcal{P}) \le f(w(\mathcal{P}))$ via an algorithm that only uses the information in $G_{\mathcal{P}}$.*

If we did not already have Theorem 7.24 then we could have used Corollary 7.34 to obtain *some* bound on $w^r(\mathcal{P})$ in terms of $w(\mathcal{P})$. The function $f(w)$ obtained in the proof of Corollary 7.34 is rather complicated and grows faster than $\frac{5^w-1}{4}$, though it is bounded by an exponential. Hence it does not offer an improvement to the bound in Theorem 7.24. However, since the recursive covering uses less information, it is an improvement in that sense.

7.5 Recursion-Theoretic Modification

By Theorem 7.14 there are recursive partial orders of width w that are not recursively w-coverable. We prove that there is always a w-covering of low degree.

Theorem 7.35 *If $\langle P, \le \rangle$ is a recursive partial order of width w, then there exists a w-covering of low degree.*

Proof:

Assume, without loss of generality, that $P = \mathsf{N}$ (but of course $\le$ has no relation to $\le^{\mathsf{N}}$). Consider the following recursive w-ary tree: The vertex $\sigma = (a_1, \ldots, a_n)$ is on T iff

1. For all i, $1 \le i \le n$ we have $1 \le a_i \le w$.

2. the map that sends i to a_i is a w-covering of $\langle P, \le \rangle$ restricted to $\{1, \ldots, n\}$.

We have (1) T is recursive, (2) T is recursively bounded by the function $f(n) = \langle w, \ldots, w \rangle$ (w appears n times), (3) any infinite branch T is a w-covering of $\langle P, \le \rangle$, (4) every w-covering of $\langle P, \le \rangle$ is represented by some infinite branch of T, and (5) the set of infinite branches of T is nonempty (by the classical Dilworth's Theorem and the previous item). Since the branches of T form a nonempty Π^0_1 class, by Theorem 3.12 there exists an infinite low branch. This branch represents a covering of low degree. ∎

7.6 Recursion-Combinatorial Modification

We now consider an 'effective version' of Dilworth's theorem which is true. The modification is both recursion-theoretic and combinatorial. It is not quite as effective as we might like: while it shows that (under certain conditions) a partial order of width w has a recursive w-covering, the proof is not uniform. Schmerl [147] showed that the proof cannot be made uniform.

This effective version is reported without proof in [94] and credited to Schmerl. This is the first published account.

Definition 7.36 Let $\langle P, \leq \rangle$ be a partial order. Define $I : P \times P \to 2^P$ via $I(x, y) = \{z \in P \mid x \leq z \leq y\}$. ($I$ stands for In between.)

Definition 7.37 Let $\langle P, \leq \rangle$ be a partial order. $\langle P, \leq \rangle$ is *locally finite* if for all x, y the set $I(x, y)$ is finite. $\langle P, \leq \rangle$ is *recursively locally finite* if it is a locally finite recursive partial order, and the function I is recursive. We abbreviate 'recursive locally finite partial order' by 'r.l.f.p.o', and 'recursively locally finite' by 'r.l.f.' Indices for r.l.f.p.o's can easily be defined.

Note that the above definition is equivalent to being able to recursively find $|I(x, y)|$.

Theorem 7.38 *If $\mathcal{P} = \langle P, \leq \rangle$ is a r.l.f.p.o then $w(\mathcal{P}) = w^r(\mathcal{P})$.*

Proof:

We prove this theorem by induction on $w = w(\mathcal{P})$. For $w = 1$ the theorem is trivial. Assume it holds for all $\mathcal{P}$ such that $w(\mathcal{P}) \leq w - 1$. Let $\mathcal{P} = \langle P, \leq \rangle$ be a recursive locally finite partial order of width w. Let $\{a_1, \ldots, a_w\}$ be a w-antichain in $\mathcal{P}$. (One cause of the proof being nonuniform is that we need to find a w-antichain. If one allows w as a parameter to an alleged uniform algorithm then w then this step would not cause non-uniformity.) Let

$$\begin{aligned} P_1 = &\quad P \cap \{x \mid (\exists i) x \geq a_i\} \\ P_2 = &\quad P \cap \{x \mid (\exists i) x \leq a_i\} \end{aligned}$$

Let $\mathcal{P}_1 = \langle P_1, \leq \rangle$ and $\mathcal{P}_2 = \langle P_2, \leq \rangle$. We will recursively w-cover $\mathcal{P}_1$ and $\mathcal{P}_2$, and then combine these w-coverings into a recursive w-covering of $\mathcal{P}$ (this combining is easy and hence omitted). The advantage of working with

$\mathcal{P}_1$ ($\mathcal{P}_2$) instead of $\mathcal{P}$ is that $\mathcal{P}_1$ ($\mathcal{P}_2$) has no infinite descending (ascending) chains.

We describe how to recursively w-cover $\mathcal{P}_1$. The w-covering of $\mathcal{P}_2$ is similar.

Let $\mathcal{A}$ be the set of all w-antichains of $\mathcal{P}_1$ (note that $\mathcal{A}$, suitably coded, is recursive). A chain C is *saturated* if for every $A \in \mathcal{A}$, $A \cap C \neq \emptyset$ (since A is an antichain and C is a chain, $|A \cap C| = 1$). It is clear that if C is a recursive chain which is saturated then $\mathcal{P}' = \langle P_1 - C, \leq \rangle$ is r.l.f and $w(\mathcal{P}') = w - 1$. Hence it suffices to construct a recursive chain C which is saturated, and then use the induction hypothesis on $\mathcal{P}'$.

We define a recursive partial order on $\mathcal{A}$ as follows:

$$A \leq B \text{ iff } (\forall a \in A)(\exists b \in B)[a \leq b]$$

(as usual $A < B$ means $A \leq B$ and $A \neq B$). We define a binary operation *glb* on pairs of elements of $\mathcal{A}$. We will later see that $glb(A, B) \leq A, B$ and no antichain that is larger has this property (so $glb(A, B)$ is the greatest lower bound of $\{A, B\}$). If $A, B \in \mathcal{A}$ then

$$glb(A, B) = \{x \in A \cup B : (\forall y \in A \cup B)[x \text{ comparable to } y \Rightarrow x \leq y]\}$$

We show that $glb(A, B) \in \mathcal{A}$, i.e., $glb(A, B)$ is an antichain of size w. Clearly $glb(A, B)$ is an antichain. Hence $|glb(A, B)| \leq w$. We show that $|glb(A, B)| \geq w$. Let $\overline{glb(A, B)}$ denote $(A \cup B) - glb(A, B)$. Note that $A \cap B \subseteq glb(A, B)$ and that $(A \cap B) \cup \overline{glb(A, B)}$ is an antichain, so it has $\leq w$ elements. Using this we have the following.

$$\begin{aligned}
|glb(A, B)| + |\overline{glb(A, B)}| &= |A \cup B| \\
|glb(A, B)| + |\overline{glb(A, B)}| &= 2w - |A \cap B| \\
|glb(A, B)| &= 2w - |A \cap B| - |\overline{glb(A, B)}| \\
|glb(A, B)| &= 2w - |(A \cap B) \cup \overline{glb(A, B)}| \\
& \text{(above line uses } A \cap B \subseteq glb(A, B)) \\
|glb(A, B)| &\geq 2w - w = w \text{ (use } |(A \cap B) \cup \overline{glb(A, B)}| \leq w).
\end{aligned}$$

Using that $\mathcal{P}_1$ is r.l.f and has no infinite descending chains one can show the following.

1. Let $A, B \in \mathcal{A}$. $glb(A, B) \leq A, B$. For all D, if $D \leq A, B$ then $D \leq glb(A, B)$. (I.e., $glb(A, B)$ is the greatest lower bound of A, B.)

2. Let $B \in \mathcal{A}$. Let $LESS(B) = \{X \in \mathcal{A} \mid X \leq B\}$. This set is finite. Moreover, given $B \in \mathcal{A}$ one can effectively find $LESS(B)$ (suitably coded).

3. Let $B \in \mathcal{A}$. The set $LESS(B)$ has a *unique* minimal element. That is, there exists *exactly one* antichain $A \in LESS(B)$ such that for all $D \in LESS(B)$ $D \not< A$. (Proof: If A and D are both minimal then note $glb(A, D) \leq A, D$ and $glb(A, D) \in LESS(B)$.) One can easily find the minimal antichain by brute force and ii.

4. Let $\mathcal{A}' \subseteq \mathcal{A}$ be such that if $A, D \in \mathcal{A}'$ then $glb(A, D) \in \mathcal{A}'$. There exists a unique minimal antichain in $\mathcal{A}'$. Moreover, given an index for $\mathcal{A}'$, one can effectively find that antichain (unless $\mathcal{A}' = \emptyset$ in which case the algorithm diverges). To do this find some $B \in \mathcal{A}'$ and then use iii. (Proof of uniqueness: if A, D are both minimal then note $glb(A, D) \leq A, D$ and $glb(A, D) \in \mathcal{A}'$.)

Using item iv the following function from $2^{\mathcal{A}}$ to $\mathcal{A}$ is well defined and can be partially computed via indices:

$$less(\mathcal{A}') = \begin{cases} \text{the min. antichain in } \mathcal{A}' & \text{if } \mathcal{A}' \text{ is closed under } glb \text{ and } \mathcal{A}' \neq \emptyset; \\ \uparrow & \text{otherwise.} \end{cases}$$

If $less(\mathcal{A}') \downarrow$ then we say that $less(\mathcal{A}')$ *exists*.

We construct a recursive saturated chain in stages. The construction might get stuck in a stage forever. If so then the chain constructed will still be saturated, and will be finite (hence recursive). We cannot tell which case occurs hence we cannot (from this proof) obtain an index for the set of elements in the chain. This is why the proof is nonuniform.

Let C_s denote the finite chain constructed by the end of stage s. Let $\mathcal{A}_s = \{A \in \mathcal{A} \mid A \cap C_s = \emptyset\}$. It is easy to see that $\mathcal{A}_s$ is recursive and that, given C_s and an index for $\mathcal{A}$, we can effectively obtain an index for $\mathcal{A}_s$. For all $s \geq 0$ C_s and $\mathcal{A}_s$ will satisfy the following

1. $(\forall A' \in \mathcal{A})[A' \cap C_s = \emptyset \text{ iff } A' \in \mathcal{A}_s]$,

2. $less(\mathcal{A}_{s-1}) \notin \mathcal{A}_s$ (for $s \geq 1$),

3. $(\forall a' \in \bigcup \mathcal{A}_s)(\forall c' \in C_s)[a' | c' \vee a' > c']$,

4. $A, D \in \mathcal{A}_s \Rightarrow glb(A, D) \in \mathcal{A}_s$ (hence if $\mathcal{A}_s \neq \emptyset$ then $less(\mathcal{A}_s)$ exists),

5. if $\mathcal{A}_s \neq \emptyset$ then $less(\mathcal{A}_s) \in \mathcal{A}_s$.

CONSTRUCTION

Stage 0: $C_0 = \{a_1\}$ (recall that a_1 was defined when $\mathcal{P}_1, \mathcal{P}_2$ were defined), $\mathcal{A}_0 = \mathcal{A} - \{A \mid a_1 \in A\}$. Clearly (1), (2), (3), (4), and (5) hold for $s = 0$.

Stage $s+1$: Find $A = less(\mathcal{A}_s)$. (If $\mathcal{A}_s \neq \emptyset$ then by (4) and (5) $less(\mathcal{A}_s)$ exists; if $\mathcal{A}_s = \emptyset$ then this computation will not halt. This is why the proof is nonuniform.) Let c be the $\leq$-largest element of C_s. By (3) we have $(\forall a \in A)[a|c \vee a > c]$. If $(\forall a \in A)[a|c]$ then $A \cup \{c\}$ would be a $(w+1)$-antichain, so $(\exists a \in A)[a > c]$. Let $C_{s+1} = C_s \cup \{a\}$, and note that $\mathcal{A}_{s+1} = \mathcal{A}_s - \{A' \mid a \in A'\}$.

END OF CONSTRUCTION

If the construction gets stuck during stage $s+1$ then let $C = C_s$, else let $C = \bigcup_s C_s$.

It is clear that (1), (2), (4) and (5) hold at the end of stage $s+1$. We show by induction on s that (3) holds. Clearly (3) holds for $s = 0$. Assume (3) is true for s but not for $s+1$. Hence $(\exists a' \in \bigcup \mathcal{A}_{s+1})(\exists c' \in C_{s+1})[a' \leq c']$. Since $\mathcal{A}_{s+1} \subseteq \mathcal{A}_s$, $(\exists a' \in \bigcup \mathcal{A}_s)(\exists c' \in C_{s+1})[a' \leq c']$. Let A, a be as in the construction during stage $s+1$. Note that $a \neq a'$ by the construction. Since (3) holds for s, the value of c' must be in $C_{s+1} - C_s = \{a\}$, hence $c' = a$. So $(\exists a' \in \bigcup \mathcal{A}_s)[a' < a]$. Let $A_1 \in \mathcal{A}_s$ be such that $a' \in A_1$. Let $A_2 = glb(A, A_1)$. Note that $A_2 \in \mathcal{A}_s$. Since $a, a' \in A \cup A_1$ and $a' < a$, $a \notin A_2$, so $A_2 \neq A$. Hence $A_2 < A$, which contradicts $A = less(\mathcal{A}_s)$.

We show that C is saturated. We need some auxiliary notions. For $m \geq 1$ let $F_m = \{A \in \mathcal{A} : |LESS(A)| = m\}$. We show that, for every $m \geq 1$, F_m is finite. For $m = 1$ note that F_1 consists of the unique minimal element of $\mathcal{A}$, so $|F_1| = 1$ and the claim is true. Let $m \geq 2$. For every $A \in F_m$ there exists $B \in LESS(A) \subseteq \bigcup_{1 \leq i < m} F_i$ that is right below A, i.e., there is no D such that $B < D < A$ (if not then $LESS(A)$ is infinite). Since $\bigcup_{1 \leq i < m} F_i$ is finite (by the induction hypothesis) and the number of A that are right below a particular element of $\mathcal{A}$ is finite (by local finiteness), the number of elements in F_m is finite.

We show, by induction on m, that for every $A \in F_m$, $A \cap C \neq \emptyset$. For $m = 1$ this is clear since the minimal antichain of $\mathcal{A}$ is $\{a_1, \ldots, a_w\}$, which intersects C_0. Let $m > 1$, and let $A \in F_m$. Let s be the least stage such that all antichains in $\bigcup_{1\le i<m} F_i$ intersect C_s. (s exists by the induction hypothesis and the finiteness of $\bigcup_{1\le i<m} F_i$.) If $A \notin \mathcal{A}_s$ then $A\cap C_s \neq \emptyset$ and if $A\in\mathcal{A}_s$ then $A = less(\mathcal{A}_s)$ so $A\cap C_{s+1}\neq\emptyset$. In either case $A\cap C_s\neq\emptyset$. ∎

The question arises as to whether the proof of Theorem 7.38 can be made uniform. Schmerl [147] showed that, in a strong sense, it cannot.

Definition 7.39 A partial order $\mathcal{P} = \langle P, \le\rangle$ is a *strongly recursively locally finite partial order* (abbreviated s.r.l.f.p.o) if its is a r.l.f.p.o and the following functions from P to N are recursive.

$$up(x) = \begin{cases} 0 & \text{if } |\{y : x\le y\}| = \infty; \\ |\{y : x \le y\}| & \text{otherwise.}\end{cases}$$

$$down(x) = \begin{cases} 0 & \text{if } |\{y : y\le x\}| = \infty; \\ |\{y : y \le x\}| & \text{otherwise.}\end{cases}$$

An index for a s.r.l.f.p.o can easily be defined. $\mathcal{P}_e$ is the s.r.l.f.p.o that is associated to index e.

Schmerl showed that even if the index of a s.r.l.f.p.o of width 2 is given, one cannot uniformly find an index for a recursive 2-covering.

Theorem 7.40 *There does not exist an algorithm $\mathcal{A}$ that, on input e an index for a s.r.l.f.p.o. of width 2, will output an index for a 2-covering of $\mathcal{P}_e$.*

Proof:

Assume that such an $\mathcal{A}$ exists. We construct a s.r.l.f.p.o $\mathcal{P}_e$ such that $\mathcal{A}(e)$ is not an index for a recursive 2-covering of $\mathcal{P}_e$.

By the recursion theorem we can assume that the construction may use e, an index for the s.r.l.f.p.o being constructed. Let $i=\mathcal{A}(e)$.

CONSTRUCTION

Stage 0: Initially the base set is $\{0,1,2\}$. The elements 0 and 1 are incomparable, and 2 is greater than both 0 and 1. Set $DIAG = FALSE$ (we have not diagonalized against $\{i\}$ yet) and $TOP = 2$.

Stage s+1: Place the least unused number u directly above TOP and then set $TOP = u$. If $DIAG = FALSE$ then run $\{i\}(0)$, $\{i\}(1)$ and $\{i\}(2)$ for s steps. If all three halt then set $DIAG = TRUE$ and do the following: if

$\{i\}(0) = \{i\}(2)$ then place the least unused number above 0 and incomparable to everything else, otherwise place the least unused number above 1 and incomparable to everything else.

END OF CONSTRUCTION.

It is easy to see that the construction yields a s.r.l.f.p.o of width 2 that has index e, but $\mathcal{A}(e)$ is not a recursive 2-covering of it.

$\blacksquare$

7.7 Miscellaneous

We have been concerned with the width of a partial order. Other parameters of partial orders (and recursive partial orders) have also been examined. We state several theorems along these lines. No proofs are given.

7.7.1 Recursive Dimension

A *realizer* of a partial order $\langle P, \leq^P \rangle$ is a set of linear orders $L_1, \ldots, L_d$ such that each one uses P as its base set, and $x \leq^P y$ iff $(\forall i) x < y$ in L_i. The *dimension* of a partial order is the minimal number of linear orders in a realizer. (An alternative definition of dimension is the least d such that P can be embedded in Q^d where Q is the rationals.) The notions of *recursive realizer* and *recursive dimension* can be defined easily.

It is known that the dimension of a partial order is $\leq$ its width. Is this true for recursive dimension and recursive width? Kierstead, McNulty and Trotter [99] have shown that this is false, but for low widths, bounds on the recursive dimension can be obtained. They showed that if $\mathcal{P}$ is a recursive partial order then (1) if $w^r(\mathcal{P}) \leq 2$, the recursive dimension of $\mathcal{P}$ is ≤ 5, (2) if $w^r(\mathcal{P}) \leq 3$, the recursive dimension of $\mathcal{P}$ is ≤ 6 (this is tight— there exists a recursive partial order $\mathcal{P}$ with $w^r(\mathcal{P}) = 3$ and recursive dimension 6), (3) there is a partial order $\mathcal{P}$ with $w^r(\mathcal{P}) = 4$ which has no finite recursive dimension ($\mathcal{P}$ also has width 3).

If we impose conditions on $\mathcal{P}$, then better bounds can be obtained. Let $\mathcal{Q}$ be the order on four elements $\{a, b, c, d\}$ where $a < b$, $c < d$, and no other pairs of elements are comparable. An *interval order* [3] is an order that does not have Q as an induced suborder (alternatively, an interval order is

[3] Interval Orders were named by Fishburn in [54] but were known to Norbert Weiner.

formed by taking the base set to be a set $\{I_1, I_2, \ldots\}$ of open intervals of reals and declare $I_i < I_j$ iff every element in I_i is less than every element in I_j). Hopkins [84] showed that if $\mathcal{P}$ is a recursive interval order of width w then (1) if $w = 2$, $\mathcal{P}$ has recursive dimension ≤ 3, (2) $\mathcal{P}$ has recursive dimension $\leq 4w - 4$; however, for all $w \geq 2$, there exists a recursive interval order $\mathcal{P}$ of width w that has recursive dimension $\lceil \frac{4}{3}w \rceil$. If the recursive width is bounded, then there are different results. Kierstead et al. [99] have shown that, for recursive interval orders, if the recursive width is $\leq w$ then the recursive dimension is $\leq 2w$.

A *Crown* is a partial order on $\{a_1, \ldots, a_n, b_1, \ldots, b_n\}$ ($n \geq 3$) such that (1) for all $i \leq n$, $a_i < b_i$, (2) for all $i \leq n-1$ $a_{i+1} < b_i$, (3) $a_1 < b_n$, and (4) no other relation exists between the elements. A partial order is *crown-free* if none of its induced suborders are crowns. Kierstead et al. [99] have shown that (1) every crown-free recursive partially ordered set with recursive width w has recursive dimension $\leq w!$, and (2) For $w \geq 3$ there is a recursive crown-free ordered set with recursive width w, width w, but recursive dimension at least $w\binom{w}{t}$ where $t = \lfloor \frac{(w-1)}{2} \rfloor$. Combining the former result with Theorem 7.24 yields that every crown-free recursive partially ordered set of width w has recursive dimension $\leq (\frac{5^w-1}{4})!$.

7.7.2 Improving the Recursive Width

Theorem 7.24 states that a recursive partial order of width w has recursive width $\leq \frac{5^w-1}{4}$. If further restrictions are made on $\mathcal{P}$, then this can be improved. Kierstead and Trotter [102] showed that if $\mathcal{P}$ is an interval order of width w then it has recursive width $\leq 3w-2$ (and this covering can be found from the index of $\mathcal{P}$). They also showed that this bound is tight— there are recursive interval orders of width w that have recursive width exactly $3w-2$. Kierstead et al. [99] showed that if P has width w and recursive dimension d, then the recursive width of P is $\leq \binom{w+1}{2}^{d-1}$.

7.7.3 Height

It is easy to show that if a partial order has height h then it can be covered by h antichains. Is this true recursively? Schmerl proved (reported in [97]) that it is not, but that a combinatorial modification is true. In particular he showed that every recursive partial order of height h can be covered by $\binom{h+1}{2}$

recursive antichains, but there are recursive partial orders of height h that cannot be covered by $\binom{h+1}{2} - 1$ recursive antichains. Bounding the recursive dimension does not help: Szeméredi and Trotter showed (reported in [97]) that there exist recursive partial orders of height h and recursive dimension 2 which cannot be covered by $\leq \binom{h+1}{2} - 1$ recursive antichains. The proof we presented of Theorem 7.14 is based on this proof.

Every height-h recursive partial order has can be covered by h low antichains. The proof uses the Low Basis Theorem (Theorem 3.12).

8 Miscellaneous Results in Recursive Combinatorics

We state several results in recursive combinatorics without proof.

8.1 Extending Partial Orders

It is easy to show that any finite partial order $\langle P, \leq \rangle$ has an extension to a linear ordering. In fact, it can even be done efficiently in $O(|P| + |\leq|)$ time [105]. A compactness argument (similar to Theorems 4.3, 5.3, 6.5, and 7.3) shows that this is true for countable partial orders. Perhaps surprisingly, a recursive analogue is *true*, that is, given an index for a recursive partial order one can effectively find an index for a linear extension of it.

Case[33] studied r.e. partial orders. He showed that the r.e. analogue is *false*, that is, there are r.e. orders $\langle P, \leq \rangle$ (both P and the set of ordered pairs $\leq$ are r.e.) that have no r.e. linear extensions. Moreover, he showed that, given any infinite r.e. set A, there is an r.e. partial order $\leq$ on A such that there are no r.e. linear extensions of $\langle A, \leq \rangle$. Roy [144] proved that every recursive partial order has a recursive linear extension and, independent of Case, also proved that there is an r.e. partial order with no r.e. linear extension.

8.2 Vizing's Theorem

An *edge k-coloring* of a graph G is a k-coloring of the edges such that no two incident edges have the same color. The *edge chromatic number of G*, denoted $\eta(G)$, is the least k such that G is edge k-colorable. Recursive edge-colorability and $\eta^r(G)$ are defined in the obvious way.

Vizing [167] ([18] is a more readily available source) showed that if G has maximal degree d then $\eta(G) \leq d+1$. His proof applied only to finite graphs, but by the usual compactness arguments (similar to Theorems 4.3, 5.3, 6.5, and 7.3) it also holds for infinite graphs. There has not been much work done on recursion-theoretic versions of Vizing's Theorem, however Kierstead [95] has shown that if G is a highly recursive graph then $\eta^r(G) \leq \eta(G)+1$. This yields a combinatorial modification of Vizing's theorem, namely that if G is highly recursive and has maximum degree d then $\eta^r(G) \leq d+2$.

8.3 Graph Isomorphism and Recursive Categoricity

Two graphs are *recursively isomorphic* if there exists a recursive isomorphism between them. A recursive graph G is *recursively categorical* if, for every G' isomorphic to G, G' is actually recursively isomorphic to G. The corresponding notions for highly recursive graphs are defined similarly. Recursive categoricity of models has been extensively studied; see [42].

It is an open problem to determine which (highly) recursive graphs are recursively categorical. Gasarch, Kueker, and Mount [64] have solved the problem for connected highly recursive rooted graphs (i.e., graphs with a distinguished vertex).

Definition 8.1 Let $G = (V, E)$ be a graph such that every vertex has finite degree. An *automorphism of* G is a map $\pi : V \to V$ that is an isomorphism of G onto itself. $Aut(G)$ is the set of automorphisms of G. $NUMAUT_G$ is the function that, on input of a nonempty finite function $X \subseteq V \times V$ and a finite sequence of elements of $x_1, \ldots, x_n \in V$, outputs the number $|\{(\pi(x_1), \ldots, \pi(x_n)) : \pi \in Aut(G), \pi \text{ extends } X\}|$. Since every vertex of G has finite degree, and X is nonempty, this number is finite.

Gasarch, Kueker, and Mount [64] showed that if G is a connected highly recursive rooted graph then G is recursively categorical iff $NUMAUT_G$ is recursive.

8.4 Eulerian and Hamiltonian Paths

Definition 8.2 Let $G = (V, E)$ be a graph with $V \subseteq \mathsf{N}$. A *path* in G is a sequence $v_1, v_2, \ldots$ such that for every $i \geq 1$, $\{v_i, v_{i+1}\} \in E$. An *Eulerian* (*Hamiltonian*) *path* is a path that uses every edge in E (vertex in V) exactly once. A *recursive Eulerian (Hamiltonian) path* is an Eulerian (Hamiltonian) path $v_1, v_2, \ldots$ such that there exists a total recursive function f with $f(i) = v_i$. A graph is called *Eulerian (Hamiltonian)* if it has an Eulerian (Hamiltonian) path[4].

Bean [11] showed that there exist Eulerian (Hamiltonian) recursive graphs with no *recursive* Eulerian (Hamiltonian) paths. For highly recursive graphs the scenario changes dramatically. Bean[11] showed that every Eulerian highly recursive graph *does* have a recursive Eulerian path; moreover, one can effectively find an index for the path given an index for the graph. However, Bean also showed that there are Hamiltonian highly recursive graphs that have no recursive Hamiltonian paths.

Beigel and Gasarch [12], and Harel [77] have studied the complexity of determining if a recursive or highly recursive graph has a (recursive) Eulerian or Hamiltonian path. Beigel and Gasarch showed (1) the problem of determining if a recursive graph has a recursive Eulerian (Hamiltonian) path is Σ_3-complete , (2) the same holds for highly recursive graphs, and (3) the problem of determining if a recursive graph has an Eulerian path is Π_3-hard and is in Σ_4 (its exact complexity is not known), (4) the problem of determining if a highly recursive graph has an Eulerian path is in Π_2 and is both Σ_1-hard and Π_1-hard. Harel showed that the problem of determining if a (highly) recursive graph has a Hamiltonian path is Σ_1^1-complete. (This implies that the problem is not in the arithmetic hierarchy.) This is only one of two results in recursive combinatorics whose complexity is outside the arithmetic hierarchy (see Section 8.13 for the other).

The vast difference between determining if a recursive graph has an Eulerian path, and determining if a recursive graph has a Hamiltonian path, might be related to the fact that the Eulerian path problem is in P, while the Hamiltonian path problem is NP-complete (see [61] for a discussion of these concepts). An open problem is to make that analogy rigorous.

[4] The definition of Eulerian (Hamiltonian) graph is nonstandard. Usually the graph is finite and is required to have an Eulerian (Hamiltonian) cycle, i.e., a path that starts at the same vertex where it ends.

8.5 Van Der Waerden's Theorem

Van der Waerden's theorem [165] states[5] that if $A \subseteq \mathsf{N}$ then either A or $\overline{A}$ has arbitrarily long arithmetic progressions. As an easy corollary either A or $\overline{A}$ has, for each k, an infinite number of arithmetic progressions of length k. Consider the weaker statement that either A has arbitrarily long arithmetic progressions or $\overline{A}$ has, for each k, an infinite number of arithmetic progressions of length k. Jockusch and Kalantari [89] considered the following 'r.e. version' of the statement: "if A is r.e. then either A has arbitrarily long arithmetic progressions, or there is an r.e. subset of $\overline{A}$ that has, for each k, an infinite number of arithmetic progressions of length k." They showed that this statement is false, but a finite form of it is true. In particular they showed the following.
(1) There exists an r.e. set A such that a) A has no arithmetic progressions of length 3, and b) no r.e. subset of $\overline{A}$ has, for each k, an infinite number of arithmetic progressions of length k.
(2) For every r.e. set A, either a) A has arbitrarily long arithmetic progressions, or b) for every k there is an r.e subset of $\overline{A}$ that has an infinite number of arithmetic progressions of length k.

Gasarch[62] investigated van der Waerden's theorem in a different way. If c is a 2-coloring of N, then a *sequence function for c* is a function that maps k to the ordered pair (a, d) such that there is a k-long monochromatic arithmetic sequence starting at a with difference d. If c is recursive, then there is a recursive sequence function by just looking for an arithmetic sequence until you find one (such will exist by van der Waerden's theorem). He posed the following conjecture: *If $\mathbf{a}$ is a nonrecursive Turing degree then there exists a coloring $c \in \mathbf{a}$ such that c has no recursive sequence function.* It is easy to show that all weakly 1-generic sets [112] are 2-colorings that satisfy the conjecture, hence the conjecture is true for weakly 1-generic degrees. If the conjecture holds for $\mathbf{a}$, then it holds for all $\mathbf{b}$ such that $\mathbf{a} \leq_{\mathrm{T}} \mathbf{b}$. Hence the conjecture is true for every degree above some weakly 1-generic degree. This includes the 1-generic sets and the n-r.e. sets. These results were proven directly in [62] without using weak 1-genericity (it is easier to use weak 1-genericity).

[5] Our formulation is equivalent to the $c = 2$ case of the standard formulation: for every c and k there exists an n such that if you c-color $\{1, \ldots, n\}$ then there is a monochromatic arithmetic progression of length k.

8.6 Sets of Positive Density

A set A has *positive upper density* if $\lim_{n\to\infty} \frac{1}{n}|A \cap \{1, \ldots, n\}| > 0$. It is easy to show that for all sets $A \subseteq \mathsf{N}$ either A or $\overline{A}$ has positive upper density. Consider the following 'r.e. version' of this statement: "if A is r.e. then either A has positive upper density or there is an r.e. subset of $\overline{A}$ that has positive upper density." Jockusch (personal communication) has shown that this statement is false. Let A be a simple set of upper density 0 (which is easily seen to exist by replacing the bound $2e$ by e^2 in Post's simple set construction in [159]). Then, since all r.e. sets disjoint from A are finite, neither A nor any r.e. subset of $\overline{A}$ has positive density.

8.7 Abstract Constructions in Recursive Graph Theory

In virtually all the proofs in recursive graph theory the recursion theory part is 'easy' and the combinatorics is 'hard' or 'clever'. Carstens and Pappinghaus [31] isolated the recursion theory from the combinatorics by proving a general theorem from which, given the proper graph-theoretic constructions, theorems from recursive graph theory can be obtained. They give three examples of theorems that can be obtained in their framework: (1) for all $d \geq 3$ there exists a connected highly recursive graph that is d-colorable but not recursively d-colorable (originally proved in [10]), (2) for all $d \geq 2$ there exists a highly recursive d-regular bipartite graph (all vertices have degree d) which has no recursive solution (originally proven in [119]), (3) for every $g \geq 1$ there exists a connected highly recursive graph of genus g that cannot be recursively embedded on an orientable surface of genus g (this seems to be new in [31]).

We suspect that the strengthening of (1) that we presented in Theorem 5.30 can be obtained in their framework.

8.8 Relativized results

Carstens [27] considered relativized versions of several of the results stated here. Instead of recursive graphs (bipartite graphs, partitions) he considered **a**-recursive graphs, where V and E are recursive in **a** (**a**-recursive bipartite graphs, etc.). All the negative results relativize easily (e.g., there exists a highly **a**-recursive graph which is k-colorable but not **a**-recursively k-colorable). For the positive results he used the relativized version of the Jockusch-Soare low basis theorem (Theorem 3.13).

8.9 Applications to Complexity Theory

Carstens and Pappinghaus [32] use recursive graph theory to show that certain types of algorithms ('extendible algorithms') will not work on several finite problems. The problems considered are matching, maxflow, and integer programming.

8.10 Applications using Σ_1^1-completeness

David Harel and Tirza Hirst [83] have been working on connecting recursive combinatorics with finite optimization problems. Given an optimization problem A that is based on an NP problem, they have set up a way to define a related problem A^+ in recursive combinatorics. Thus, for example, the infinite version of maximum-clique becomes the question of whether a recursive graph has an infinite clique.

They have shown that if $A^+ \notin \Pi_0^2$ then $A \notin$ Max-NP, and hence $A \notin$ Max-SNP either (see [130] for definitions). They also have a general result that makes it possible to "lift up" certain NP reductions to become Σ_1^1 reductions. The enables one to prove that, for some A's, the recursive counterpart A^+ is Σ_1^1-complete; hence $A^+ \notin \Pi_0^2$, and so $A \notin$ Max-NP. These two results provide a framework for proving that certain optimization problems are outside Max-NP and Max-SNP.

Arora et al. [6] have shown that, unless P=NP, problems that are hard for the class Max-SNP by a certain kind of approximation-preserving reduction, cannot be approximated by a polynomial-time approximation scheme unless P=NP (see their paper for exact definitions). The results of Harel and Hirst show that certain problems are not directly subject to this bad news. Of course, these problems may still be hard to approximate, but the techniques of [6] are probably not able to establish this.

Harel and Hirst [78] have used these two results to prove that many problems in recursive combinatorics are Σ_1^1-complete. Here is a partial list of the finitary versions, which, as explained, are therefore all outside Max-NP and Max-SNP:

1. maximum-clique (this was known to be outside Max-SNP [7]),
2. max-independent-set (this is essentially the same as max-clique),
3. max-Hamiltonian-path (in the sense that we seek the path with maximum tag, where we tag a path by k if it covers the first k nodes of the graph in some fixed ordering),
4. max-set-packing (i.e., the maximal number of nonoverlapping sets from among a given collection of sets),
5. complement of min-vertex-cover (this is really the same as max-clique),
6. max-subgraph (given graphs H and G, find the subgraph of H with maximal tag that is part of G; here again we tag a subgraph with k if it covers the first k nodes),
7. complement of min-set-cover,
8. largest common subsequence (given a set of strings, find largest string that is a – perhaps noncontinuous – substring of each string in the set),
9. max-color (very much like max-independent-set),
10. max-exact-cover (even restricted to sets with 3 elements),
11. max-domino or max-tiling (maximum k for which the $k \times k$ subgrid of an $n \times m$ grid is tileable).

8.11 Ramsey-Type Edge Colorings

In this section we do not restrict edge colorings as we did in Section 8.2.

Definition 8.3 A graph is i-connected if removing any $i-1$ vertices leaves it connected. Let Γ_3 be the set of 3-connected graphs unioned with the triangle graph.

Let $H_1, H_2 \in \Gamma_3$. Burr [23] showed that it is undecidable if a given partial (finite) edge coloring of a highly recursive graph can be extended to a coloring c such that there are no RED H_1's or BLUE H_2's. (He actually used a much more restrictive notion than highly recursive.)

Gasarch and Grant [63] showed that there are highly recursive graphs that can be edge colored in a triangle-free manner, but not recursively so colored. They also showed that determining if a particular graph can be recursively colored in a triangle-free manner is Σ_3-complete.

8.12 Schröder-Bernstein Theorem and Banach's Theorem

The Schröder-Bernstein theorem[6] states that if there exist injections $f : A \to B$ and $g : B \to A$ then there exists a bijection $h : A \to B$. Banach [9] refined this theorem by showing that if there exist injections $f : A \to B$ and $g : B \to A$ then there exist partitions $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$ such that f restricted to A_1 is a bijection between A_1 and B_1, and g^{-1} restricted to A_2 is a bijection from A_2 to B_2 (in short, $f[A_1] \cup g^{-1}[A_2]$ is a bijection of A onto B).

Remmel [138] showed that the recursive analogue of the Schröder-Bernstein theorem holds, but the recursive analogue of Banach's theorem does not. For both recursive versions the premise is that there exist partial injections f, g and recursive sets A, B such that f (g) is defined on all of A (B). From this, the existence of a partial recursive bijection from A to B (defined on all of A) is easily shown. However, Remmel constructed f, g, A, B such that no recursive A_1, A_2, B_1, B_2 (as in Banach's theorem) exist.

8.13 König's Max-Min Theorem

Let $G = (A, B, E)$ be a finite bipartite graph. A *matching* is a set of disjoint edges. A *cover* is a set of vertices C such that every edge contains a vertex in C. The *matching number* is the maximum cardinality of a matching. The *covering number* is the minimal cardinality of a cover.

König ([107], see [116] for a modern version in English) showed that the matching number and covering number are identical. Lovász and Plummer [116] consider this to be the most important theorem in matching theory. We consider König's matching theorem for countable bipartite graphs. To

[6] Schröder announced the theorem in 1896, but his proof was flawed (see [108] for the full story). Bernstein published the first correct proof in 1898 in [19]. Cantor also had a proof, but it used the axiom of choice, which was not needed.

give the statement substance we need the following special type of cover. A cover C of G is a *König cover* if there exists a matching M such that C can be obtained by picking one vertex from every edge in M.

Aharoni [1, 2] showed that every bipartite graph has a König cover. Aharoni, Magidor, and Shore [3] investigated this theorem in terms of both proof theory and recursion theory. They showed that (1) compactness, or König's Lemma on infinite trees, is not enough, from a proof-theoretic viewpoint, to prove the theorem, (2) there exist recursive bipartite graphs such that all König covers are of degree above all the hyperarithmetic Turing degrees, and (3) for every recursive bipartite graph there exists a König cover of degree $\leq_{\mathrm{T}} \mathcal{O}$ where $\mathcal{O}$ is Kleene's O (of degree Σ_1^1).

8.14 Arrow's Theorem

Let V be a finite set which we think of as being individuals (or voters). Let X be a finite set which we think of as alternatives being decided upon by the society of individuals (perhaps by voting). Let P be a subset of all rankings of X which we think of as the orders on X that are allowed to be chosen. Let G be a function that takes the information consisting of every individual's preferred ranking of X (these rankings must be in P) and outputs a ranking in P. The tuple (V, X, P, G) is called a *society* and is intended to model how a society chooses among alternatives.

Arrow [8] showed that a set of four reasonable conditions on a society imply that there exists a 'dictator', i.e., an individual $v \in V$ such that G will rank X the same way v does. Skala [156] showed that the infinite version of Arrow's theorem depends on the model of set theory. In particular, if ZF is consistent, then there is a model of ZF+AC where Arrow's theorem is false for countable V; however, if one assume the Axiom of Determinacy then there is a model where Arrow's theorem is true for countable V.

Since the classic Arrow's theorem is not true for countable V, recursive combinatorics will play a different role than usual. In this context it is used to recover *some* version of Arrow's theorem that is true. Lewis [114] defined r.e. society, recursive society, and recursive dictator functions. He has shown that an r.e. version of Arrow's theorem, with countable V, is true; and that a recursive version of Arrow's theorem, with countable V, is true with a primitive recursive dictator function.

See [113] and [125] more on this topic.

8.15 An Undecidable Problem in Finite Graph Theory

Let G be a finite graph,v be a vertex of G, and $r \in \mathsf{N}$. The *r-neighborhood* of v is the induced subgraph with vertex set consisting of all vertices of distance at most r from v. Let r-*neib*(G) be the set of all r-neighborhoods of a graph G. Note that r-*neib*(G) is a set of graphs. Consider the following problem: given a finite set of graphs $\{H_1, \ldots, H_k\}$, and a number $r \in \mathsf{N}$, does there exist a graph G such that r-*neib*$(G) = \{H_1, \ldots, H_k\}$? Winkler[171] has shown that this problem is undecidable in general. However, if the cycle length of G is bounded then the problem is solvable. This result can be used to solve the following problem: given k and a finite set $D \subseteq \mathsf{N}$, does there exist a k-ary tree whose degree set is D? This problem had been solved earlier by Winkler [170].

8.16 Hindman's Theorem

Hindman [81] proved the following remarkable theorem: If c is a k-coloring of N, then there exists an infinite monochromatic set X such that every sum of elements from X is the same color. We call such a set *sum-homogeneous*.

Blass, Hirst, and Simpson [17] have analyzed this theorem recursion-theoretically. They have shown (1) there exists a recursive 2-coloring of N such that for all sum-homogeneous sets X, $X \not\leq_{\mathrm{T}} K$, (2) for all k-colorings c of N there exists a sum-homogeneous set that is recursive in $\emptyset^{\omega+1}$.

8.17 Recursive Linear Orderings

A *recursive linear ordering* (henceforth RLO) is a linear ordering where the order relation is recursive. For information on both the classic and recursive theories of linear orderings, see For a survey of recursive linear orderings see [49]. We give one example of a line of research in this area which fits into our theme.

It is a classic theorem that if L is an infinite linear order then it has either an infinite ascending or infinite descending suborder. Tennenbaum (see [143]) has shown that this theorem is false recursively, that is, there exist infinite RLO's with no r.e. suborder isomorphic to either ω or ω^* (ω^* is the order $\ldots, 3, 2, 1, 0$). Tennenbaum's order is isomorphic to $\omega^* + \omega$. Watnick [169] characterized exactly which order types may have RLO's that are recursive counterexamples. Let $\mathcal{L}$ be the set of all such order types. He showed that $L \in \mathcal{L}$ iff $L \cong \omega + Z\alpha + \omega^*$ where α is a Π_2 linear order (Π_2 base set and Π_2 relation).

8.17.1 Recursive Automorphisms

It is easy to see that the order $\omega^* + \omega$ has a non-trivial automorphism. Moses [127] has shown that this is not true recursively. He has shown that an RLO L has a nontrivial automorphism iff L has a dense suborder.

8.18 Well Quasi Orderings

A *quasi-order* $\mathcal{P}$ is a set P (called the *base set*) together with a relation $\leq$ that is transitive and reflexive, but not necessarily anti-symmetric. (e.g., take $P = \{0,1\}^\omega$ and $\leq$ is subsequence). A *well-quasi-order* (henceforth wqo) is a quasi order $\langle P, \leq \rangle$ with the following additional property: if $p_1, p_2, \ldots$ is an infinite sequence of elements from P then there exists $i < j$ such that $p_i \leq p_j$.

Kruskal [110] showed that the set of trees, ordered via homeomorphic embedding (or minor), form a wqo (see [128, 129] for an elegant proof). Robertson and Seymour [141] have shown the far more difficult result that the set of all graphs, ordered under minors, is wqo. Another interesting example of a wqo is Σ^* (where Σ is a finite alphabet) under subsequence (proof uses similar techniques to those in [128, 129]).

What makes wqo's interesting is the following theorem: If $\langle P, \leq \rangle$ is a wqo and $Q \subseteq P$ is closed downward under $\leq$ then there exists a finite number of elements $p_1, \ldots, p_k \in P$ such that

$$Q = \{q \in P : \bigwedge_{i=1}^{k} p_i \not\leq q\}.$$

The set $\{p_1, \ldots, p_k\}$ is called the *obstruction set for Q*. For example, since the set of graphs of genus $\leq g$ (some fixed g) is closed under minors, and graphs under minor is a wqo, for every g there is a finite obstruction set $\mathcal{O}_g$ such that G has genus g iff G does not have an element of $\mathcal{O}_g$ as a minor. For genus 1 (planar) the obstruction set is known to be $\{K_5, K_{3,3}\}$. For another example, let Σ be any finite alphabet and let $X \subseteq \Sigma^*$. Let $SUBSEQ(X)$

be the set of all subsequences of strings in X. Since Σ^* under subsequence is a wqo, and $SUBSEQ(X)$ is closed under subsequence, there is a finite obstruction set for $SUBSEQ(X)$. This implies the (somewhat remarkable) theorem that if X is *any* language whatsoever then the set of subsequences of X is regular. Kruskal [111] notes that this was first proven (using different terminology) by Higman [80] and has been proven several times since then, most recently by Haines [73].

The proof that sets closed downward under $\leq$ have finite obstruction sets is not hard, but it is noneffective. In [15] the recursive analogue is considered and shown to be false.

Harvey Friedman has shown that finite versions of Kruskal's Theorem are unprovable in Peano Arithmetic. See [153] or [115] for a proof, and see [158] for an exposition. Friedman, Robertson, and Seymour have examined proof theoretic considerations of the Graph Minor Theorem [55].

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