Pythagorean Ramsey Theory Assignment
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1 Pythagorean Ramsey Theorem

Notation 1.1 If $n \in \mathsf{N}^+$ then $[n] = \{1, \ldots, n\}$.

Def 1.2 Let $n, c \in \mathsf{N}^+$. Let COL: $[n] \rightarrow [c]$.

1. A *Mono Pythagorean Triple* is $x < y < z \in [n]$ such that
   (a) $\mathrm{COL}(x) = \mathrm{COL}(y) = \mathrm{COL}(z)$, and
   (b) $x^2 + y^2 = z^2$.
2. COL is a *proper coloring* if there are no mono Pythagorean triples.

The following was proved by Heule & Kullman & Marek in 2016.

Theorem 1.3

1. *There is a 2-coloring of* [7824] *has no mono Pythagorean triples (so there is a proper 2-coloring of* [7824]*).*
2. *For all 2-coloring of* [7825] *there is a Mono Pythagorean Triples (so there is no proper 2-coloring of* [7825]*).*

The proof was done by a SAT solver and the full proof is about 200 terabytes.

2 What Can We Do Without Those Resources

Since there is a proper 2-coloring of [7824] there is a proper 2-coloring of much shorter segments of N. We want to find proper 2-colorings of larger and larger $[n]$ *using simple algorithms and an ordinary computer (like a laptop).*

Here is a simple greedy algorithm which also prints out comments, helpful for debugging.

```
COL[1] = 1.
For z = 2 to ∞ (the program will quit at some point)
      (We want to set COL(z) = 1 but will see if we can)
      1FINE=TRUE (initially think COL(z) = 1 is fine)
      2FINE=TRUE (initially think COL(z) = 2 is fine)
      For x = 1 to z
        For y = 1 to z
          If x^2 + y^2 = z^2 THEN
            IF COL(x) = COL(y) = 1 then
                1FINE=FALSE
                PRINT((x, y) makes 1FINE=FALSE).
            IF COL(x) = COL(y) = 2 then
                2FINE=FALSE
                PRINT((x, y) makes 2FINE=FALSE).
      If 1FINE=TRUE then COL(z) = 1
      else
        If 2FINE=TRUE then COL(z) = 2
      else
        PRINT(CANNOT COLOR PAST z − 1)
        Jump out of loop and end program.
```

3 Randomized Greedy

RAND$(1, 2)$ means that the computer picks one of 1,2 at random.
COL[1] $= 1$.

For $z = 2$ to ∞ (the program will quit at some point)

```
    (We want to set COL(z) = 1 but will see if we can)
    1FINE=TRUE (initially think COL(z) = 1 is fine)
    2FINE=TRUE (initially think COL(z) = 2 is fine)
    For x = 1 to z
      For y = 1 to z
        If x^2 + y^2 = z^2 THEN
          IF COL(x) = COL(y) = 1 then
              1FINE=FALSE
              PRINT((x, y) makes 1FINE=FALSE).
          IF COL(x) = COL(y) = 2 then
              2FINE=FALSE
              PRINT((x, y) makes 2FINE=FALSE).
    If 1FINE=TRUE and 2FINE=TRUE then COL(z) = RAND(1, 2)
    else
      If 1FINE=TRUE and 2FINE=FALSE then COL(z) = 1
    else
    If 1FINE=FALSE and 2FINE=TRUE then COL(z) = 2
    else
      PRINT(CANNOT COLOR PAST z − 1)
      Jump out of loop and end program.
```