Asy Lower Bounds on Ramsey Numbers

Exposition by William Gasarch

Summary Of Talk

- We obtain asy lower bounds on $R(k)$.

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- We then use the **method** to do other things, outside of Ramsey Theory.

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We want to find **lower bounds**

PROBLEM We want to **find** a coloring of the edges of K_n w/o a mono K_k. for some $n = f(k)$.

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Theorem $R(k) \geq (k-1)^2$.

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Here is a coloring of the edges of $K_{(k-1)^2}$ with no mono K_k:
First partition $[(k-1)^2]$ into $k-1$ groups of $k-1$ each.

$$COL(x,y) = \begin{cases} \mathrm{RED} & \text{if } x,y \text{ are in same } V_i \\ \mathrm{BLUE} & \text{if } x,y \text{ are in different } V_i \end{cases} \quad (1)$$

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- They can't all be in one V_i, so it can't have RED K_k.

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Look at any k vertices.

- They can't all be in one V_i, so it can't have RED K_k.
- They can't all be in different V_i, so it can't have BLUE K_k.

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PROBLEM We want to **find** a coloring of the edges of K_n without a mono K_k for some $n \geq k^2$.

WRONG QUESTION I only need show that such a coloring **exists**.

Pick a coloring at Random!

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$$\binom{n}{k} \times 2 \times 2^{\binom{n}{2}-\binom{k}{2}}$$

Prob that a random 2-coloring HAS a homog set is bounded by

$$\frac{\binom{n}{k} \times 2 \times 2^{\binom{n}{2}-\binom{k}{2}}}{2^{\binom{n}{2}}} \leq \frac{\binom{n}{k} \times 2}{2^{\binom{k}{2}}} \leq \frac{n^k}{k!2^{k(k-1)/2}}$$

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Prob that the coloring HAS a homog set of size k is $\le \frac{n^k}{k!2^{k(k-1)/2}}$.

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Recap If we color $\binom{[n]}{2}$ at random then

Prob that the coloring HAS a homog set of size k is $\le \frac{n^k}{k!2^{k(k-1)/2}}$.

IF this prob is < 1 then **there exists** a coloring of the edges $\binom{[n]}{2}$ with **no homog set of size** k.

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So if $\frac{n^k}{k!2^{k(k-1)/2}} < 1$ then **there exists** a coloring of the edges $\binom{[n]}{2}$ with **no homog set of size** k.

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We will work out the algebra of $\frac{n^k}{k!2^{k(k-1)/2}} < 1$ on the next slide; however, the real innovation here is that we show that a coloring exists by showing that the prob that it does not exists is < 1.

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Working Out the Inequality

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Stirling's Fml $k! \sim (2\pi k)^{1/2} \left(\frac{k}{e}\right)^k$, so $(k!)^{1/k} \sim (2\pi k)^{1/2k} \left(\frac{k}{e}\right)$

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Want n large. $n = \frac{1}{e\sqrt{2}} k 2^{k/2}$ works.

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Joel Spencer told me he was hoping for a better improvement.

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- I would **not** call the Prob Method and application of Ramsey. (Some articles do.)
- I would say that Ramsey Theory was the initial motivation for the Prob Method which is now used for many other things, some of which are practical.

DISTINCT DIFF SETS

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Can we do better?
STUDENTS break into small groups and try to either do better OR show that you best you can do is $O(\log n)$.

An Approach

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KEY: If the prob is strictly greater than 0 then there must be SOME set of a elements where all of the diffs are distinct.

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If you pick a RANDOM $A \subseteq \{1, \ldots, n\}$ of size a what is the probability that all of the diffs in A are NOT distinct?

We hope the Prob is strictly LESS THAN 1.

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We only need to show that the prob is LESS THAN 1.

Review a Little Bit of Combinatorics

The number of ways to CHOOSE y elements out of x elements is

$$\binom{x}{y} = \frac{x!}{y!(x-y)!}.$$

Determining the Prob I

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Two ways to create a set with a diff repeated:

Way One:

- Pick $x < y$. There are $\binom{n}{2} \le n^2$ ways to do that.
- Pick diff d such that $x + d \ne y$, $x + d \le n$, $y + d \le n$. Can do $\le n$ ways. Put $x, y, x + d, y + d$ into A.
- Pick $a - 4$ more elements out of the $n - 4$ left.

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Number of ways to do this is $\le n^3 \times \binom{n-4}{a-4}$.

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Way Two: Pick $x < y$. Let $d = y - x$ (so we do NOT pick d). Put $x, y = x + d, y + d$ into A. Pick $a - 3$ more elements out of the $n - 3$ left.

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Determining the Prob II

If you pick a RANDOM $A \subseteq \{1, \ldots, n\}$ of size a then a bound on the probability that all of the diffs in A are NOT distinct is

$$\frac{n^3 \times \binom{n-4}{a-4} + n^2 \times \binom{n-3}{a-3}}{\binom{n}{a}} = \frac{n^3 \times \binom{n-4}{a-4}}{\binom{n}{a}} + \frac{n^2 \times \binom{n-3}{a-3}}{\binom{n}{a}}$$

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$$= \frac{n^3 a(a-1)(a-2)(a-3)}{n(n-1)(n-2)(n-3)} + \frac{n^2 a(a-1)(a-2)}{n(n-1)(n-2)}$$

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$$\leq \frac{32a^4}{n}$$ Need some Elem Algebra and uses $n \geq 5$.

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UPSHOT: For all $n \ge 5$ there exists a all-diff-distinct subset of $\{1, \ldots, n\}$ of size roughly $n^{1/4}$.

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- Caveat: If the Prob Proof has high prob of getting the object, then seems constructive. If all you prove is nonzero, than maybe not.

Actually Can Do Better

- With a maximal set argument can do $\Omega(n^{1/3})$.
- Better is known: $\Omega(n^{1/2})$ which is optimal.
  (That is a result by Kolmos-Sulyok-Szemeredi from 1975)

SUM FREE SET PROBLEM

Exposition by William Gasarch

Sum Free Set Problem

A More Sophisticated Use of Prob Method.

Definition: A set of numbers A is *sum free* if there is NO $x, y, z \in A$ such that $x + y = z$.

Example: Let $y_1, \ldots, y_m \in (1/3, 2/3)$ (so they are all between $1/3$ and $2/3$). Note that $y_i + y_j > 2/3$, hence $y_i + y_j \notin \{y_1, \ldots, y_m\}$.

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Proof: STUDENTS DO THIS. ITS EASY.
Example: Let $A = \{y_1, \ldots, y_m\}$ all have fractional part in $(1/3, 2/3)$. A is sum free by above Lemma.

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STUDENTS - WORK ON THIS IN GROUPS.

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$$B_a = \{x \in A : \mathrm{frac}(ax) \in (1/3, 2/3)\}.$$

For all a, B_a is sum-free by Lemma above.
SO we need an a such that B_a is LARGE.

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We take $U - L$ large enough so that this prob is $\geq (1/3 - \epsilon)$.

$$E(|B_a|) = \sum_{x\in A} \Pr_{a\in[L,U]}(\mathrm{frac}(ax) \in (1/3, 2/3))$$
$$= \sum_{x\in A}(1/3 - \epsilon)$$
$$= (1/3 - \epsilon)n.$$

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$$\begin{aligned} E(|B_a|) &= \sum_{x \in A} \mathrm{Pr}_{a \in [L,U]}(\mathrm{frac}(ax) \in (1/3, 2/3)) \\ &= \sum_{x \in A} (1/3 - \epsilon) \\ &= (1/3 - \epsilon)n. \end{aligned}$$

So THERE EXISTS an a such that $|B_a| \geq (1/3 - \epsilon)n$.
What is a? I DON"T KNOW AND I DON"T CARE!
End of Proof

Turan’s Theorem

Exposition by William Gasarch

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more easily using Probability, but first need a lemma. The proof

we give is due to Ravi Boppana and appears in the Alon-Spencer book on *The Probabilistic Method*

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Proof: Try to count the edges by summing the degrees at each vertex. This counts every edge TWICE.

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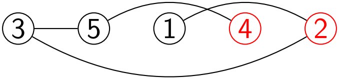

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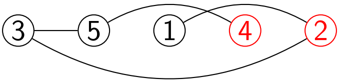

The set of vertices that have NO edges coming out on the right form an Ind Set. Call this set I.

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WRONG QUESTION!

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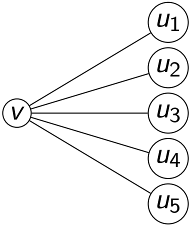
What is the EXPECTED VALUE of the size of I.

(NOTE- we permuted the vertices RANDOMLY)

What is Prob $v \in I$

Let $v \in V$. What is prob that $v \in I$



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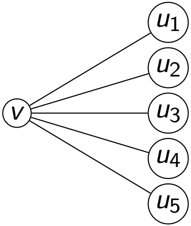

v has degree d_v. How many ways can v and its vertices be laid out: $(d_v + 1)!$. In how many of them is v on the right? $d_v!$.

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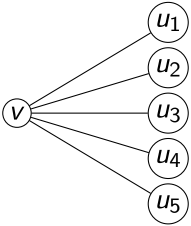

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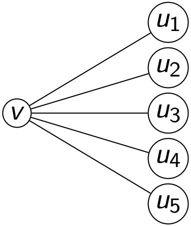

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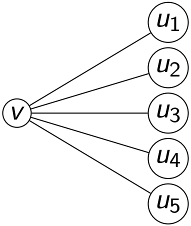

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Hence

$$E(|I|) = \sum_{v \in V} \frac{1}{d_v + 1}.$$

How Big is this Sum?

Need to find lower bound on

$$\sum_{v \in V} \frac{1}{d_v + 1}.$$

Rephrase

NEW PROBLEM:
Minimize

$$\sum_{v \in V} \frac{1}{x_v + 1}$$

relative to the constraint:

$$\sum_{v \in V} x_v = 2e.$$

KNOWN: This sum is minimized when all of the x_v are $\frac{2e}{|V|} = \frac{2e}{n}$.
So the min the sum can be is

$$\sum_{v \in V} \frac{1}{\frac{2e}{n} + 1} = \frac{n}{\frac{2e}{n} + 1}.$$

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$$E(I) \ge \sum_{v \in V} \frac{1}{x_v + 1} \ge \sum_{v \in V} \frac{1}{\frac{2e}{n} + 1} = \frac{n}{\frac{2e}{n} + 1}.$$

END OF THIS TALK/TAKEAWAY

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TAKEAWAY: There are TWO ways (probably more) to show that an object exists using probability.

1. Show that the probability that it exists is NONZERO. Hence there must be some set of random choices that makes it exist. We did this for the distinct-sums problem.
2. You want to show that an object of a size $\geq s$ exists. Show that if you do a probabilistic experiment then you (a) always get the object of the type you want, and (b) the expected size is $\geq s$. Hence again SOME set of random choices produces an object of size $\geq s$.