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Shelah's proof of the Hales–Jewett theorem revisited

Pierre Matet

Université de Caen-CNRS, Mathématiques, BP 5186, 14032 Caen Cedex, France

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Abstract

We present a variant of Shelah's proof of the Hales–Jewett theorem.

In [1] Graham, Rothschild and Spencer compile a list of six major theorems in Ramsey theory. These "super six" are (in the order of [1]) Ramsey's theorem, van der Waerden's theorem, Schur's theorem, Rado's theorem, the Hales–Jewett theorem and the Graham–Leeb–Rothschild theorem. Some of these results are related. For instance Schur's theorem, which easily follows from Ramsey's theorem (for pairs), is actually a special case of Rado's theorem. As for van der Waerden's theorem, it may be obtained as a corollary of the Hales–Jewett theorem. In the words of [1], "the Hales–Jewett theorem strips van der Waerden's theorem of its unessential elements and reveals the heart of Ramsey theory. It provides a focal point from which many results can be derived and acts as a cornerstone for much of the more advanced work". The original proof of the Hales–Jewett theorem [2] proceeded by double induction (on the number c of colors and the size n of the monochromatic set). Shelah's celebrated proof of the theorem [3] uses simple induction (on n). It gives primitive recursive bounds for the Hales–Jewett theorem (and thus also for van der Waerden's theorem). Our proof follows that of Shelah. Simply, we replace what is sometimes called Shelah's pigeonhole lemma (and is proved in [3] by a brute force argument) by an appeal to Ramsey's theorem. We hope that this version of the proof will make the Hales–Jewett theorem more accessible. Let us remark that our argument, like Shelah's, provides a primitive recursive bound for the Hales–Jewett function.

We start with some notation. Given two sets a and b, ${}^{a}b$ denotes the set of all functions from a to b. We adopt the set-theoretic convention that each nonnegative integer n is the set of all

E-mail address: matet@math.unicaen.fr.

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nonnegative integers $m < n$. Given two positive integers ℓ and n, $G(\ell, n)$ denotes the set of all functions g such that $\mathrm{dom}(g) \subsetneq \ell$ and $\mathrm{ran}(g) \subseteq n$. For $g \in G(\ell, n)$, $X(\ell, n, g)$ denotes the set of all $x \in {}^{\ell}n$ such that $x \restriction \mathrm{dom}(g) = g$ and x is constant on $l \setminus \mathrm{dom}(g)$.

The Hales–Jewett theorem asserts that given two positive integers n and c, there exists a positive integer ℓ with the property that for every $V : {}^{\ell}n \longrightarrow c$, there is $g \in G(\ell, n)$ such that V is constant on $X(\ell, n, g)$. We let $HJ(n, c)$ denote the least such ℓ.

Given a set b and a nonnegative integer r, $[b]^r$ denotes the set of all size r subsets of b.

Ramsey's theorem states that given three positive integers t, r and k with $t \geq r$, there exists an integer $m \geq t$ with the property that for every $F : [m]^r \longrightarrow k$, there is $a \in [m]^t$ such that F is constant on $[a]^r$. We let $R_k(t, r)$ denote the least such m.

To make our proof of the Hales–Jewett theorem easier to follow, we will first see how it works in the simplest nontrivial case, that is when $n = 3$ and $c = 2$. Thus let $W : {}^{h}3 \to 2$, where h is some fixed positive integer. For $0 \leq a_1 < a_2 < a_3 < a_4 \leq h$, define $f_{a_1a_2a_3a_4} : {}^{2}3 \to {}^{h}3$ by

$$f_{a_1a_2a_3a_4}(y) : p \to \begin{cases} 1 & \text{if } 0 \leq p < a_1 \\ y(0) & \text{if } a_1 \leq p < a_2 \\ 2 & \text{if } a_2 \leq p < a_3 \\ y(1) & \text{if } a_3 \leq p < a_4 \\ 1 & \text{if } a_4 \leq p < h. \end{cases}$$

For $y \in {}^{2}3$, define $\hat{y} \in {}^{2}2$ by $\hat{y}(i) = \min\{y(i), 1\}$. Suppose for a while that for any $y \in {}^{2}3$, $W(f_{a_1a_2a_3a_4}(y)) = W(f_{a_1a_2a_3a_4}(\hat{y}))$. Then we are done. In fact, since $HJ(2, 2) = 2$, we can find $g \in G(2, 2)$ and $m < 2$ so that $W \circ f_{a_1a_2a_3a_4}$ is constantly m on $X(2, 2, g)$. Then clearly, $W \circ f_{a_1a_2a_3a_4}$ is constantly m on $X(2, 3, g)$.

To find (a_1, a_2, a_3, a_4) as desired, we proceed as follows. For $\alpha = 0, 1, 2$, let

$$s_\alpha : \{(v, x, z) : 0 \leq v < x < z \leq h\} \to {}^{h}3$$

be defined by

$$s_\alpha(v, x, z) : p \longmapsto \begin{cases} 1 & \text{if } 0 \leq p < v \\ 2 & \text{if } v \leq p < x \\ \alpha & \text{if } x \leq p < z \\ 1 & \text{if } z \leq p < h. \end{cases}$$

Furthermore for $\beta = 0, 1$, let

$$t_\beta : \{(v, x, z) : 0 \leq v < x < z \leq h\} \to {}^{h}3$$

be defined by

$$t_\beta(v, x, z) : p \longmapsto \begin{cases} 1 & \text{if } 0 \leq p < v \\ \beta & \text{if } v \leq p < x \\ 2 & \text{if } x \leq p < z \\ 1 & \text{if } z \leq p < h. \end{cases}$$

Note that

$$f_{a_1a_2a_3a_4}(y) = s_{y(1)}(a_{3-y(0)}, a_3, a_4)$$

if $y(0) \in \{1, 2\}$, and

$$f_{a_1 a_2 a_3 a_4}(y) = t_{y(0)}(a_1, a_2, a_{y(1)+2})$$

if $y(1) \in \{1, 2\}$ and $y(0) \neq 2$.

Define

$$F : \{(v, x, z) : 0 \leq v < x < z \leq h\} \to 2 \times 2 \times 2 \times 2 \times 2$$

by

$$F(v, x, z) = ((W \circ s_0)(v, x, z), (W \circ s_1)(v, x, z), (W \circ s_2)(v, x, z), (W \circ t_1)(v, x, z), (W \circ t_2)(v, x, z)).$$

Now if $h+1 \geq R_{32}(4, 3)$, we can find $a \in [h+1]^4$ so that F is constant on $\{(v, x, z) \in a \times a \times a : v < x < z\}$. Let $a_1 < a_2 < a_3 < a_4$ be the elements of a. Then (a_1, a_2, a_3, a_4) is as desired.

To prove the Hales–Jewett theorem in full generality, we fix c and proceed by induction on n. It is easy to see that $HJ(1, c) = 1$. Given $\ell = HJ(n, c)$, set $w = (n + 1)^\ell - n^\ell$, $k = c^w$ and $q = R_k(2\ell, 2\ell - 1)$. We will show that $HJ(n + 1, c) < q$.

Let D be the set of all $d \in {}^{(2\ell+2)}q$ such that

$$0 = d(0) \leq d(1) \leq \cdots \leq d(2\ell) \leq d(2\ell + 1) = q - 1.$$

Define $f : {}^{\ell}(n + 1) \times D \longrightarrow {}^{(q-1)}(n + 1)$ so that for each $j \leq 2\ell$, $f(y, d)$ is constant on the set $\{v : d(j) \leq v < d(j + 1)\}$ with value $n - 1$ if $j \equiv 0 \bmod 4$, n if $j \equiv 2 \bmod 4$, and $y(\frac{j-1}{2})$ otherwise. For $0 \leq i < \ell$, let Y_i be the set of all $y \in {}^{\ell}(n + 1)$ such that $y(i) = n$ and $y(j) < n$ for every $j < i$, and define $s_i : Y_i \times [q]^{2\ell-1} \longrightarrow {}^{(q-1)}(n + 1)$ by $s_i(y, b) = f(y, d_{b,i})$, where $d_{b,i}$ is the unique element d of D such that $b = \{d(j) : 1 \leq j \leq 2\ell\}$ and $d(2i + 1) = d(2i + 2)$. Note that the set $\bigcup_{0 \leq i < \ell} Y_i$ has cardinality $\sum_{i=0}^{\ell-1}(n^i \cdot (n + 1)^{\ell-1-i}) = (n + 1)^\ell - n^\ell$.

Now fix $W : {}^{(q-1)}(n + 1) \longrightarrow c$. Define $F : [q]^{2\ell-1} \longrightarrow {}^{(\bigcup_{0 \leq i < \ell} Y_i)}c$ so that $(F(b))(y) = W(s_i(y, b))$ whenever $y \in Y_i$. There is $a \in [q]^{2\ell}$ such that F is constant on $[a]^{2\ell-1}$. Let $e \in D$ be such that $a = \{e(j) : 1 \leq j \leq 2\ell\}$.

For $y \in {}^{\ell}(n + 1)$, define $\hat{y} \in {}^{\ell}n$ by $\hat{y}(i) = \min\{y(i), n - 1\}$. We claim that $W(f(y, e)) = W(f(\hat{y}, e))$. Suppose $u \neq \phi$, where $u = \{i < \ell : y(i) = n\}$, and let $u_1, \ldots, u_r$ be the increasing enumeration of the elements of u. Set $y_0 = y$ and define $y_1, \ldots, y_r \in {}^{\ell}(n + 1)$ so that for $1 \leq j \leq r$, $y_j(u_j) = n - 1$ and y_j and y_{j-1} agree on $\ell \backslash \{u_j\}$. Note that $y_r = \hat{y}$. For $1 \leq j \leq r$,

$$\{f(y_{j-1}, e), f(y_j, e)\} = \{s_{u_j}(y_{j-1}, a \backslash \{e(2u_j + 1)\}), s_{u_j}(y_{j-1}, a \backslash \{e(2u_j + 2)\})\}$$

(with $s_{u_j}(y_{j-1}, a \setminus \{e(2u_j + 1)\})$ being equal to $f(y_j, e)$ if u_j is even, and to $f(y_{j-1}, e)$ otherwise), so $W(f(y_{j-1}, e)) = W(f(y_j, e))$. It follows that $W(f(y_0, e)) = W(f(y_r, e))$.

Now define $V : {}^{\ell}n \longrightarrow c$ by $V(y) = W(f(y, e))$. Select $g \in G(\ell, n)$ and $m < c$ so that V takes the constant value m on $X(\ell, n, g)$. Then for every $y \in X(\ell, n + 1, g)$, $W(f(y, e)) = W(f(\hat{y}, e)) = V(\hat{y}) = m$, so we are done.

Note that the proof gives $HJ(2, c) < c + 1$ (which is optimal).

It should be clear from the proof that the full strength of Ramsey's theorem is not needed.

For positive integers k and ℓ, let $S_k(\ell)$ denote the least $m \geq 2\ell$ such that for any $F : [m]^{2\ell-1} \to k$, there is $a \in [m]^{2\ell}$ with the property that for every $i < \ell$, $F(a \setminus \{e(2i + 1)\}) = F(a \setminus \{e(2i + 2)\})$, where $e(1), \ldots, e(2\ell)$ is the increasing enumeration of the elements of a.

The proof above shows that $HJ(n+1,c) < S_k(\ell)$, where $\ell = HJ(n,c)$ and $k = c^{((n+1)^\ell - n^\ell)}$. It is simple to see that $S_k(1) = k+1$. To conclude this paper, let us show that $S_k(\ell+1) \leq S_{k'}(\ell) + k + 1$, where $k' = k^{\binom{k+1}{2}}$. Thus let $K : [m']^{2\ell+1} \to k$, where $m' = S_{k'}(\ell) + k + 1$. Set $Z = \{z : S'_k(\ell) \leq z < m'\}$. Define $F : [S_k(\ell)]^{2\ell-1} \to {}^{[Z]^2}k$ by $F(d) : v \longmapsto K(d \cup v)$. Pick $b \in [S_k(\ell)]^{2\ell}$ so that for any $r < \ell$, $F(b \setminus \{e(2r+1)\}) = F(b \setminus \{e(2r+2)\})$, where $e(1), \ldots, e(2\ell)$ is the increasing enumeration of the elements of b. There must be $S_{k'}(\ell) \leq x < y < m'$ such that $K(b \cup \{x\}) = K(b \cup \{y\})$. Now set $a = b \cup \{x, y\}$, $e(2\ell+1) = x$ and $e(2\ell+2) = y$. Then for any $i < \ell + 1$, $K(a \setminus \{e(2i+1)\}) = K(a \setminus \{e(2i+2)\})$.

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