

## The Infinite Ramsey Theorem and the Large Ramsey Theorem

### 1 The Infinite Ramsey Theorem

**Def 1.1** Let  $a, c \in \mathbf{N}$ . Let  $A$  be a set ( $A$  will usually be  $\mathbf{N}$  or  $[n]$  or  $\{k, \dots, n\}$ ). Let  $COL: \binom{A}{a} \rightarrow [c]$ .  $H \subseteq A$  is *homogenous* if  $COL$  is constant on  $\binom{H}{a}$ .

In this manuscript we will only talk about 2-colorings of  $\binom{A}{2}$ . Generalizations to any number of colors are trivial. Generalizations to different values of  $a$  are fairly easy but may require some thought.

**Theorem 1.2** *Every 2-coloring  $\binom{\mathbf{N}}{2}$  has an infinite homogenous set.*

**Proof:** Let  $COL: \binom{\mathbf{N}}{2} \rightarrow [2]$ . We define an infinite sequence of vertices,

$$x_1, x_2, \dots,$$

and an infinite sequence of sets of vertices,

$$V_0, V_1, V_2, \dots,$$

that are based on  $COL$ .

Here is the intuition: Vertex  $x_1 = 1$  has an infinite number of edges coming out of it. Some are RED, and some are BLUE. Hence there are an infinite number of RED edges coming out of  $x_1$ , or there are an infinite number of BLUE edges coming out of  $x_1$  (or both). Let  $c_1$  be a color such that  $x_1$  has an infinite number of edges coming out of it that are colored  $c_1$ . Let  $V_1$  be the set of vertices  $v$  such that  $COL(\{v, x_1\}) = c_1$ . Then keep iterating this process.

We now describe it formally.

$$\begin{aligned} V_0 &= \mathbf{N} \\ x_1 &= 1 \end{aligned}$$

$$c_1 = \begin{cases} \text{RED} & \text{if } |\{v \in V_0 \mid COL(\{v, x_1\}) = \text{RED}\}| \text{ is infinite;} \\ \text{BLUE} & \text{otherwise} \end{cases} \quad (1)$$

$$V_1 = \{v \in V_0 \mid COL(\{v, x_1\}) = c_1\} \text{ (note that } |V_1| \text{ is infinite)}$$

Let  $i \geq 2$ , and assume that  $V_{i-1}$  is defined. We define  $x_i$ ,  $c_i$ , and  $V_i$ :

$$x_i = \text{the least number in } V_{i-1}$$

$$c_i = \begin{cases} \text{RED} & \text{if } |\{v \in V_{i-1} \mid COL(\{v, x_i\}) = \text{RED}\}| \text{ is infinite;} \\ \text{BLUE} & \text{otherwise} \end{cases} \quad (2)$$

$$V_i = \{v \in V_{i-1} \mid COL(\{v, x_i\}) = c_i\} \text{ (note that } |V_i| \text{ is infinite)}$$

How long can this sequence go on for? Well,  $x_i$  can be defined if  $V_{i-1}$  is nonempty. We can show by induction that, for every  $i$ ,  $V_i$  is infinite. Hence the sequence

$$x_1, x_2, \dots,$$

is infinite.

Consider the infinite sequence

$$c_1, c_2, \dots$$

Each of the colors in this sequence is either RED or BLUE. Hence there must be an infinite sequence  $i_1, i_2, \dots$  such that  $i_1 < i_2 < \dots$  and

$$c_{i_1} = c_{i_2} = \dots$$

Denote this color by  $c$ , and consider the vertices

$$H = \{x_{i_1}, x_{i_2}, \dots\}$$

It is easy to show that  $H$  is homog.

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## 2 Finite Ramsey from Infinite Ramsey

Having proved the infinite Ramsey Theorem, we then want to prove the finite Ramsey Theorem. Can we prove the finite Ramsey Theorem *from* the infinite Ramsey Theorem? Yes, we can! This proof will not give any bounds. Other proofs do.

**Theorem 2.1** *For all  $k$  there exists  $n$  such that for all  $COL: \binom{[n]}{2} \rightarrow [2]$  there exists a homog set of size  $k$ .*

**Proof:** Suppose, by way of contradiction, that there is some  $k \geq 2$  such that no such  $n$  exists. For every  $n \geq k$ , there is some way to color  $\binom{[n]}{2}$  so that there is no homog set of size  $k$ . Hence there exist the following:

1.  $COL_0$ , a 2-coloring of  $\binom{[k]}{2}$  that has no homog set of size  $k$ .
2.  $COL_1$ , a 2-coloring of  $\binom{[k+1]}{2}$  that has no homog set of size  $k$ .
3.  $COL_2$ , a 2-coloring of  $\binom{[k+2]}{2}$  that has no homog set of size  $k$ .
4.  $COL_3$ , a 2-coloring of  $\binom{[k+3]}{2}$  that has no homog set of size  $k$ .
- $\vdots$
- $j$ .  $COL_L$ , a 2-coloring of  $\binom{[k+L]}{2}$  that has no homog set of size  $k$ .
- $\vdots$

We will use these 2-colorings to form a 2-coloring  $COL$  of  $\binom{\mathbb{N}}{2}$  that has no infinite homog set. This contradicts Theorem ??.

Let  $e_1, e_2, e_3, \dots$  be a list of every element of  $\binom{\mathbb{N}}{2}$ . We will color  $e_1$ , then  $e_2$ , etc.

How should we color  $e_1$ ? We will color it the way an infinite number of the  $COL_i$ 's color it. Call that color  $c_1$ . Then how to color  $e_2$ ? Well, first consider ONLY the colorings that colored  $e_1$  with color  $c_1$ . Color  $e_2$  the way an infinite number of those colorings color it. And so forth.

We now proceed formally:

$$J_0 = \mathbb{N}$$

$$COL(e_1) = \begin{cases} \text{RED} & \text{if } |\{j \in J_0 \mid COL_j(e_1) = \text{RED}\}| \text{ is infinite;} \\ \text{BLUE} & \text{otherwise.} \end{cases} \quad (3)$$

$$J_1 = \{j \in J_0 \mid COL(e_1) = COL_j(e_1)\}$$

Let  $i \geq 2$ , and assume that  $e_1, \dots, e_{i-1}$  have been colored. Assume, furthermore, that  $J_{i-1}$  is infinite and, for every  $j \in J_{i-1}$ ,

$$\begin{aligned} COL(e_1) &= COL_j(e_1) \\ COL(e_2) &= COL_j(e_2) \\ &\vdots \\ COL(e_{i-1}) &= COL_j(e_{i-1}) \end{aligned}$$

We now color  $e_i$ :

$$COL(e_i) = \begin{cases} \text{RED} & \text{if } |\{j \in J_{i-1} \mid COL_j(e_i) = \text{RED}\}| \text{ is infinite;} \\ \text{BLUE} & \text{otherwise.} \end{cases} \quad (4)$$

$$J_i = \{j \in J_{i-1} \mid COL(e_i) = COL_j(e_i)\}$$

One can show by induction that, for every  $i$ ,  $J_i$  is infinite. Hence this process *never* stops.

**Claim:** Let  $A$  be a finite subset of  $\{k, k+1, \dots\}$ . Then there exists an infinite number of  $i$  such that  $COL$  on  $\binom{A}{2}$  agrees with  $COL_i$  on  $\binom{A}{2}$ .

**Proof of Claim**

Left to the reader.

**End of Proof of Claim**

We have produced a 2-coloring of  $\binom{\mathbb{N}}{2}$ . Let By Theorem ?? there is an infinite homog set for  $COL$ :

$$H = \{x_1 < x_2 < x_3 < \dots\}.$$

Look at

$$H' = \{x_1 < x_2 < \dots < x_k\}$$

This is a homog set with respect to  $COL$ . By the claim there is an  $i$  (in fact, infinitely many) such that  $COL$  and  $COL_i$  agree on  $\binom{H'}{2}$ . Clearly  $H'$  is a homog set of size  $k$  for  $COL_i$ . This contradicts the definition of  $COL_i$ .

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### 3 Proof of Large Ramsey Theorem

In all of the theorems presented earlier, the labels on the vertices did *not* matter. In this section, the labels *do* matter.

**Def 3.1** A finite set  $F \subseteq \mathbb{N}$  is called *large* if the size of  $F$  is BIGGER than the smallest element of  $F$ .

#### Example 3.2

1. The set  $\{1, 2, 10\}$  is large: It has 3 elements, the smallest element is 1, and  $3 > 1$ .
2. The set  $\{5, 10, 12, 17, 20\}$  is NOT large: It has 5 elements, the smallest element is 5, and 5 is NOT strictly greater than 5.
3. The set  $\{20, 30, 40, 50, 60, 70, 80, 90, 100\}$  is NOT large: It has 9 elements, the smallest element is 20, and  $9 < 20$ .
4. The set  $\{5, 30, 40, 50, 60, 70, 80, 90, 100\}$  is large: It has 9 elements, the smallest element is 5, and  $9 > 5$ .
5. The set  $\{101, \dots, 190\}$  is not large: It has 90 elements, the smallest element is 101, and  $90 < 101$ .

We will be considering monochromatic  $K_m$ 's where the underlying set of vertices is a large set. We need a definition to identify the underlying set.

Let  $COL$  be a 2-coloring of  $\binom{[n]}{2}$ . Consider the set  $\{1, 2\}$ . It is clearly both homogeneous and large (using our definition of large). Hence the statement

“for every  $n \geq 2$ , every 2-coloring of  $K_n$  has a large homogeneous set”

is true but trivial.

What if we used  $V = \{k, k + 1, \dots, n\}$  as our vertex set? Then a large homogeneous set would have to have size at least  $k$ .

**Notation 3.3**  $LR(k)$  is the least  $n$ , if it exists, such that every 2-coloring of  $\binom{\{k, \dots, n\}}{2}$  has a large homogeneous set.

**Theorem 3.4** For every  $k \geq 2$  there exists  $n$  such that for all 2-colorings of  $\binom{\{k, \dots, n\}}{2}$  there exists a large homog set.

**Proof:** This proof is similar to our proof of the finite Ramsey Theorem from the infinite Ramsey Theorem (the proof of Theorem ??).

Suppose, by way of contradiction, that there is some  $k \geq 2$  such that no such  $n$  exists. For every  $n \geq k$ , there is some way to color  $\binom{\{k, \dots, n\}}{2}$  so that there is no large homog sets. Hence there exist the following:

1.  $COL_1$ , a 2-coloring of  $\binom{\{k, k+1\}}{2}$  that has no large homog set.
2.  $COL_2$ , a 2-coloring of  $\binom{\{k, k+1, k+2\}}{2}$  that has no large homog set.
3.  $COL_3$ , a 2-coloring of  $\binom{\{k, \dots, k+3\}}{2}$  that has no large homog set.
- ⋮
- $j$ .  $COL_L$ , a 2-coloring of  $\binom{\{k, \dots, k+L\}}{2}$  that has no large homog set.
- ⋮

We will use these 2-colorings to form a 2-coloring  $COL$  of  $\binom{\{k, k+1, \dots\}}{2}$ . This coloring will have an infinite homog set by Theorem ??. This will give us a contradiction to the definition of one of the  $COL_i$ .

Let  $e_1, e_2, e_3, \dots$  be a list of every element of  $\binom{\{k, k+1, \dots\}}{2}$ . We will color  $e_1$ , then  $e_2$ , etc.

How should we color  $e_1$ ? We will color it the way an infinite number of the  $COL_i$ 's color it. Call that color  $c_1$ . Then how to color  $e_2$ ? Well, first consider ONLY the colorings that colored  $e_1$  with color  $c_1$ . Color  $e_2$  the way an infinite number of those colorings color it. And so forth.

We now proceed formally:

$$J_0 = \mathbb{N}$$

$$COL(e_1) = \begin{cases} \text{RED} & \text{if } |\{j \in J_0 \mid COL_j(e_1) = \text{RED}\}| \text{ is infinite;} \\ \text{BLUE} & \text{otherwise.} \end{cases} \quad (5)$$

$$J_1 = \{j \in J_0 \mid COL(e_1) = COL_j(e_1)\}$$

Let  $i \geq 2$ , and assume that  $e_1, \dots, e_{i-1}$  have been colored. Assume, furthermore, that  $J_{i-1}$  is infinite and, for every  $j \in J_{i-1}$ ,

$$\begin{aligned} COL(e_1) &= COL_j(e_1) \\ COL(e_2) &= COL_j(e_2) \\ &\vdots \\ COL(e_{i-1}) &= COL_j(e_{i-1}) \end{aligned}$$

We now color  $e_i$ :

$$COL(e_i) = \begin{cases} \text{RED} & \text{if } |\{j \in J_{i-1} \mid COL_j(e_i) = \text{RED}\}| \text{ is infinite;} \\ \text{BLUE} & \text{otherwise.} \end{cases} \quad (6)$$

$$J_i = \{j \in J_{i-1} \mid COL(e_i) = COL_j(e_i)\}$$

One can show by induction that, for every  $i$ ,  $J_i$  is infinite. Hence this process *never* stops.

**Claim:** Let  $A$  be a finite subset of  $\{k, k+1, \dots\}$ . Then there exists an infinite number of  $i$  such that  $COL$  on  $\binom{A}{2}$  agrees with  $COL_i$  on  $\binom{A}{2}$ .

**Proof of Claim**

Left to the reader.

**End of Proof of Claim**

By Theorem ?? there is an infinite homog set for  $COL$ :

$$H = \{x_1 < x_2 < x_3 < \dots\}.$$

Look at

$$H' = \{x_1 < x_2 < \dots < x_{x_1+1}\}$$

This is a homog set with respect to  $COL$ . By the claim there is an  $i$  (in fact, infinitely many) such that  $COL$  and  $COL_i$  agree on  $\binom{H'}{2}$ . Clearly  $H'$  is a large homog set for  $COL_i$ . This contradicts the definition of  $COL_i$ .

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